Universität Regensburg
Fakultät für Mathematik

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Instanton Gauge Theory

Exercise Sheet 1

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Exercices labeled with a star $*$ are going to be graded. Please hand in solutions to Felix Eberhart not later than Monday 10 May at noon.

Exercise 1. Let M be a smooth manifold.

(a) Show that two smooth vector bundles over M are isomorphic as bundles if and only if their spaces of sections are isomorphic as $\mathcal{C}^\infty(M)$-modules.

(b) Let $p\colon P \to M$ be a principal fibre bundle with structure group G and let G act on a set X via $\rho\colon G \to \mathrm{Aut}(X)$. Show that there is a canonical isomorphism

$$\Gamma(P \times_\rho X) \cong \{f\colon P \to X \mid f(pg) = \rho(g)f(p)\}.$$

(c) Let $p\colon E \to M$ be a vector bundle with fibre V, $\mathcal{U} = \{U_i \mid i \in I\}$ an open covering of M and for every $i \in I$ let $\phi_i\colon E\big|_{U_i} \to U_i \times V$ be an isomorphism of vector bundles. The map $\phi_i \circ \phi_j^{-1}\colon U_i \cap U_j \times V \to U_i \cap U_j \times V$ is then a morphism of vector bundles and is thus of the form $\phi_i \circ \phi_j^{-1}((u,v)) = (u, \phi_{ij}(u) \cdot v)$ where $\phi_{ij}\colon U_i \cap U_j \to Gl(V)$ is a continuous (or smooth in the case of smooth vector bundles) map. Show that that a system $(\phi_{ij}\colon U_i \cap U_j \to Gl(V))_{i,j\in I}$ arises from a vector bundle in the above way if and only if it is a *cocycle*, i.e. it has the following properties:

- $\forall i \in I\colon \phi_{ii} \equiv const_{\mathrm{id}_V}$
- $\forall i,j,k \in I\colon \phi_{ij} \circ \phi_{jk} = \phi_{ik}$

(d) Let $p\colon E \to M$ and $q\colon F \to M$ be vector bundles with fibres V and W. Then there is an open covering $(U_i \mid i \in I)$ over which both E and F can be trivialized. Choose such trivializations and let (ϕ_{ij}) be the corresponding cocycle of E and (ψ_{ij}) the cocycle of F.

(i) For $i, j \in I$ let

$$\phi_{ij} \otimes \psi_{ij}\colon U_i \cap U_j \to Gl(V \otimes W),$$
$$u \mapsto (v \otimes w \mapsto \phi_{ij}(v) \otimes \psi_{ij}(w))$$

Show that this defines a cocycle and thus a vector bundle. This bundle is denoted by $E \otimes F$ and called the *tensor product bundle*.

(ii) Show that if E and F are smooth vector bundles there is a canonical isomorphism

$$\Gamma(E \otimes F) \cong \Gamma(E) \otimes_{\mathcal{C}^\infty(M)} \Gamma(F)$$

where $\Gamma(-)$ denotes smooth sections.

(e) Let $p\colon P \to M$ be a smooth principal fibre bundle with structure group G, let $\rho\colon G \to Gl(V)$ and $\rho'\colon G \to Gl(W)$ be two smooth representations of G and let $E := P \times_\rho V$ and $F := P \times_{\rho'} W$ be the associated smooth vector bundles. Denote by

$$\rho \otimes \rho'\colon G \to Gl(V \otimes W),$$
$$g \mapsto (v \otimes w \mapsto (\rho(g) \cdot v) \otimes (\rho'(g) \cdot w))$$

the diagonal action on $V \otimes W$. Show that $P \times_{\rho \otimes \rho'} (V \otimes W)$ is isomorphic to $E \otimes F$.

Exercise 2*. Compute the parallel transport for the covariant derivative $d + a$, where $a \in \Omega^1([0,1], i\mathbb{R})$ on the trivial complex line bundle on $[0,1]$ along the path joining the point 0 to the point 1.

Exercice 3*. Denote by $\pi\colon S^3 \to S^2$ the Hopf fibration introduced in class, a principal S^1-fibre bundle. For any $n \in \mathbb{N}$, we consider the cyclic subgroup $C_n \subseteq S^1$ generated by the n^{th} roots of unity. We denote by $Q_n := S^3/C_n$ the quotient space.

(a) Show that Q_n is a principal S^1-fibre bundle over S^2 in a natural way.
(b) Let $\rho\colon S^1 \to \mathrm{Aut}(\mathbb{C})$ denote the canonical isomorphism which maps a complex number to the multiplication by that number. Determine the first Chern classes of the complex line bundles $Q_n \times_\rho \mathbb{C}$.
(c) Compare these line bundles with the tensor powers $H^{\otimes k}$ of the tautological line bundle.

Exercice 4. Solve the analogue of Exercice 3 for the general Hopf fibration $S^1 \to S^{2n+1} \to \mathbb{CP}^n$.

Exercice 5. Suppose $\pi\colon P \to M$ is a principal G-fibre bundle. Suppose P carries a Riemannian metric (suppressed from notation), and that G acts (freely) by isometries on P. Show that the orthogonal complement to the vertical tangent bundle of P is a connection.

Exercice 6. For the Hopf fibration $\pi\colon S^3 \to S^2$ the S^1-action is by isometries. Write down the connection 1-form for the connection studied in the previous exercice.

Exercice 7*. In class we have defined two $S^3 \cong SU(2)$-bundles $P_\pm$ over the quaternionic projective space $\mathbb{HP}^1$, say P_+ via the action

$$S^{4n+3} \times S^3 \to S^{4n+3}$$
$$((h_0, \dots, h_n), g) \mapsto (h_0 g, \dots, h_n g)$$

and P_- via the action

$$S^{4n+3} \times S^3 \to S^{4n+3}$$
$$((h_0, \dots, h_n), g) \mapsto (\overline{g} h_0, \dots, \overline{g} h_n)$$

Here $\overline{g}$ denotes the quaternionic conjugation of g, and S^3 is identified with the quaternions of unit length.

(a) Show that the quaternionic projective space $\mathbb{HP}^1$ is diffeomorphic to S^4.
(b) Show that the action of S^3 is by isometries in both cases.
(c) Write down the connection 1-forms of the connection given by the orthogonal complement of the vertical tangent bundle in both cases.
(d) Decide whether any of these connections is selfdual or anti-selfdual.