Instanton Gauge Theory

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<u>Rmk:</u> Uhlenbeck's fundamental lemma remains true with the same constants on a euclidean ball of arbitrary radius (bc. the L^2-norm of 2-forms in 4 dimensions is conformally invariant.)

It also remains true on balls with metric that is suff. C^2-close to the eucl. metric (with slightly adjusted constants).

<u>Lemma</u> (curvature is proper): Let B be a eucl. ball

Let $$SU_{\varepsilon_0/2} := \{ A = \Gamma + a \in L^2_1 \mid \int_B |F_A|^2 < \frac{\varepsilon_0}{2}, \; d^* a = 0, \; *a|_{\partial B} = 0 \}.$$

Then
$$F_{(-)} : SV_{\varepsilon_0/2} \to L^2(B; \Lambda^2 T^*B \otimes \mathrm{ad}\, P),$$
$$A \longmapsto F_A$$

is proper.

<u>Pf:</u> Assume (A_i) is a sequence in $SV_{\varepsilon_0/2}$ s.t. (F_{A_i}) is Cauchy. Then $(A_i = \Gamma + a_i)$

$$\int_{B^4} |F_{A_i} - F_{A_j}|^2 \overset{\text{Cauchy-Schwarz}}{\gtrsim} \int_B |d(a_i - a_j)|^2 - \int_B |(a_i - a_j)\wedge a_j + a_j \wedge (a_i - a_j)|^2$$

$$\underset{\uparrow}{\geq} \int_B |(d+d^*)(a_i - a_j)|^2 - C\|a_i - a_j\|_{L^2_1}^2 \cdot \left(\|a_i\|_{L^2_1}^2 + \|a_j\|_{L^2_1}^2\right) =: (*)$$

$d^* a_i = d^* a_j = 0$,

Hölder $\|fg\|_{L^2}^2 \leq \|f\|_{L^4}^2 \|g\|_{L^4}^2$,

$L^2_1 \hookrightarrow L^4$

Have (WF) $\Delta b = \nabla^* \nabla b + Ric(V, V)$,

where $V \in TM$ is dual to b.

From this, one can show

$$\int_B |(d+d^*)b|^2 = \int_B |\nabla_r b|^2 + \int_{\partial B} *b \wedge d^* b + \int_{\partial B} |b|^2$$

In particular,

$$\int_B |(d+d^*)b|^2 \geq \int_B |\nabla_r b|^2.$$

As last time, $\int_B |\nabla_r b|^2 \geq \lambda_1^2 \|b\|^2_{L^2(B)}$,

$\lambda_1 > 0$.

Hence, by the key Lemma from last time, we get

$$\circledast \geq C_2 \|a_i - a_j\|^2_{L^2_1} - \varepsilon C C_3 \|a_i - a_j\|^2_{L^2_1} =$$

$$= (C_2 - \varepsilon C C_3) \|a_i - a_j\|^2_{L^2_1} .$$

For $\varepsilon \ll 1$, $C_2 > \varepsilon C C_3$, so we get that (a_i) is Cauchy in $L^2_1(B)$ ☒

Local compactness

Let $r > 0$ & B_r the ball of radius r in $\mathbb{R}^4$.

Let $M^\varepsilon(B_r) := \{ [A] \mid F_A^+ = 0 \ \& \int_{B_r} |F_A|^2 < \varepsilon \}$.

By restriction, we get a map

$$\mathrm{res}: M^{\varepsilon}(B_1) \longrightarrow \mathcal{M}(B_r)$$

for any $r < 1$.

<u>Thm:</u> For ε small enough, res is compact.

$\star : \Omega^2(\mathrm{ad}P) \circlearrowleft$

$F_A^+ := F_A + \star F_A$

<u>Pf:</u> We want to use properness of the curvature map for conn. in Uhlenbeck gauge. So, given $([A_i])_i$ in $M^{\varepsilon}(B_1)$ we want to show that there is a subsequence s.t. the sequence of curvatures is L^2-convergent.

By Uhlenbeck's fund. Lemma, we can chose repr. of $([A_i])$ in Uhlenbeck gauge &

these repr. will be uniformly L^2_1-bounded $\left(\|A_i - \Gamma\|_{L^2_1} < C \cdot \|F_{A_i}\|_{L^2} \leq \varepsilon \right)$. Thus, there is an L^2_1-weakly convergent subsequ, call it (A_i) again,

$$A_i \xrightarrow{w} A$$

From this, we get

$$F_{A_i} \xrightarrow{w} F_A \quad \text{in } L^2.$$

If we can show that $\|F_{A_i}\|_{L^2} \to \|F_A\|_{L^2}$ then by a general lemma from funct. analysis,

$$F_{A_i} \xrightarrow{s} F_A$$

& we are done by properness.

We need:

Reminder: A – conn. on $SU(2)\times B^4 \to B^4$,

then $$CS(A) := -\int_B tr(F_A \wedge F_A)$$

& $\|F_A\|_{L^2}^2 = CS(A) \iff A$ is ASD

Lemma: $$CS(A) = \int_{\partial B} -tr\left(b\wedge db + \frac{2}{3} b\wedge[b\wedge b]\right),$$

$$b := (A-\Gamma)|_{\partial B}$$

Pf: If A, B are two conn. s.t. $A|_{\partial B} = B|_{\partial B}$, can glue to get conn. on the trivial bundle over S^4 & get

$$0 = CS(A \# -B) = CS(A) - CS(B).$$

$\uparrow$ Chern-Weil

Thus, $CS(A)$ only depends on $A|_{\partial B}$.

Let A ~~be~~ be ~~the~~ the conn. on $S^3 \times [0,1] \times SU(2)$

$$\downarrow$$

$$S^3 \times [0,1]$$

defined by the form $\Gamma + tb$, $b \in \Omega^1(S^3; \mathrm{ad}P)$.

Then by explicit calculation one shows

$$\int_{S^3 \times [0,1]} \mathrm{tr}(F_A \wedge F_A) = \int_{S^3} \mathrm{tr}\left(b \wedge db + \frac{2}{3} b \wedge [b \wedge b]\right),$$

By gluing in a trivial bdle-w-conn. on B^4, we get the formula for any conn. on B^4.

$\square$

Back to our sequence (A_i):

By Fubini's theorem theorem there is some $M>0$ such that for all i there is a subset $I_i \subseteq [r_1, 1]$ of positive measure s.t.

$$\| a_i|_{\partial B_{\delta_i}} \|_{L^2_1(\partial B_{\delta_i})} < M$$

$|a_i|^2 + |\nabla_r a_i|^2$ is measurable on B_δ (even integrable)

for any $\delta_i \in I_i$.

Can choose subsequence (still called (A_i)) s.t.

$\bigcap_i I_i$ has positive measure & is hence nonempty. We get uniform $L^2_1(\partial B_\delta)$-boundedness of $(a_i|_{\partial B_\delta})$ for any $\delta \in \bigcap_i I_i$

Now $L^2_1(\partial B_\delta) \hookrightarrow L^2_{1/2}(\partial B_\delta)$ is compact, So can choose a further subsequence s.t.

$$a_i|_{\partial B_\delta} \longrightarrow a|_{\partial B_\delta}$$

$L^2_{1/2}$ - strongly.

Now, as all A_i are ASD,

$$\|F_{A_i}\|^2_{L^2(B_\delta)} = CS(A_i) \overset{\text{Lemma}}{=}$$

$$= \int_{\partial B_\delta} -\operatorname{tr}\left(a_i|_{\partial B_\delta} \wedge da_i|_{\partial B_\delta} + \frac{2}{3} a_i|_{\partial B} \wedge [a_i|_{\partial B} \wedge a_i|_{\partial B}]\right).$$

But, bc. $a_i \in L^2_{1/2}(\partial B)$, $da_i \in L^2_{-1/2}(\partial B)$

$(\Rightarrow a_i \wedge da_i \in L^2)$, $w(L^2_{1/2}) = \frac{1}{2} - \frac{3}{2} = -1$

& $3 \cdot w(L^2_{1/2}) = -3 = w(L^1)$,

this is $L^2_{1/2}$ - continuous, hence

from $a_i|_{\partial B_\delta} \to a|_{\partial B_\delta}$ $L^2_{1/2}$-strongly,

we get
$$\|F_{A_i}\| \longrightarrow \|F_A\|.$$

<u>Remaining problem:</u> $(\Gamma + a_i)|_{B_\delta}$ in no longer in Uhlenbeck gauge, as $*a_i|_{\partial B_\delta} \neq 0$ in general.

But $*a_i|_{\partial B_\delta}$ is still small by continuity.

One then shows that there is a gauge transformation g_i over B_δ s.t.

1) $g_i a_i$ is in Uhlenbeck gauge on B_δ

2) g_i is "small", so g_i is in the identity component of $\mathcal{G}$

2) implies that $CS(g_i A_i) = CS(A_i)$ & by 1), we can use properness of curvature.

<u>One can do better</u>:

<u>Prop</u>: A an L^2_1-ASD-conn., $\|F_A\|_{L^2}$ sufficiently small. Then $\exists$ an L^2_2-gauge transformation g s.t. $gA \in C^\infty$.

<u>Prop</u>: In the local compactness thm, the convergent subsequence can actually be chosen up to gauge to converge in C^∞.

Uhlenbeck compactness

Thm: G cpt Lie group, $P \to X$ a principal G-bundle over a smooth, closed, Riemannian 4-mfd. Let (A_i) be a sequence of ASD connections on P with

$$\int_X |F_A|^2 \, dvol = 8\pi^2 k \quad , k \in \mathbb{Z}$$

Then after passing to a subsequence, there are

- fin. many points $x_1, \dots, x_n \in X$ $(n \leq k)$
- a G-bundle $P' \to X$
- a sequence of bundle isomorphisms

$$g_i : P|_{X \setminus \{x_1, \dots, x_n\}} \longrightarrow P'|_{X \setminus \{x_1, \dots, x_n\}}$$

• an ASD conn. A on $\mathbb{P}^1$ s.t.

$$g_i A_i \longrightarrow A$$

in C^∞ over compact subsets of

$X \setminus \{x_1, \dots, x_n\}$.

<u>Pf:</u> By Banach–Alaoglu, there is a subsequence s.t. $|F_{A_i}|^2 dvol$ converges as a measure, i.e.

$$\int_X f \, |F_{A_i}|^2 dvol \longrightarrow \int_X f \, d\nu$$

for some bounded measure ν & all $f \in C^0(X)$.

Because $\int_X |F_{A_i}|^2 dvol = 8\pi^2 k$,

$\int_X d\nu = 8\pi^2 k$.

For $\varepsilon > 0$ there are thus at most

$$\left\lfloor \frac{8\pi^2 k}{\varepsilon^2} \right\rfloor$$

points which do <u>not</u> lie in a geodesic ball of ν-measure $\leq \varepsilon^2$. By the local compactness result & local C^2-closeness of the metric on balls to the euclidean metric, we get a sequence of geodesic balls

$$\{B_\alpha\}_{\alpha \in \mathbb{N}}$$

where $\int_{B_\alpha} |F_{A_i}|^2 \, d\mathrm{vol} \leq \varepsilon^2$ for all α & $i \gg 1$

& s.t. a sequence of proper subballs

$B_\alpha' \Subset B_\alpha$ still covers all of

$X \setminus \{x_1, \dots, x_n\}$.

For $B'_\alpha \subset B_\alpha$ use local compactness result:

Up to gauge, a subsequence of (A_i) converges in $L^2_1(B'_\alpha)$. By taking a diagonal sequence we get a subsequence that converges in all of the $L^2_1(B'_\alpha)$. These local gauges $g_{i\alpha}$ can be chosen in such a way that they patch together to give bundle automorphisms of $P|_{X \setminus \{x_1, \dots, x_n\}}$, i.e. can be made to satisfy the equation

$$\varphi_{\alpha\beta} = g_{i\alpha}\big|_{B'_\alpha \cap B'_\beta} \cdot g_{i\beta}^{-1}\big|_{B'_\alpha \cap B'_\beta}$$

where $\varphi_{\alpha\beta}$ is the cocycle obtained by local trivializations of P over B_α' (hard!).

By glueing these local bundles-with-connection, we get an ASD conn. A on $P|_{X\setminus\{x_1,\dots,x_n\}}$ s.t.

$$g_i A_i \longrightarrow A$$

in C^∞ on cpt subsets of $X \setminus \{x_1,\dots,x_n\}$.

Now we need:

<u>Theorem</u> (Uhlenbeck's removable singularities thm)

If A is an ASD-conn. ~~order~~ $B^4\setminus\{0\}$ with $\|F_A\|_{L^2(B^4\setminus\{0\})} < \infty$

then there is a gauge transformation g over $B^4\setminus\{0\}$ & an ASD conn. A' on B^4

s.t.

$$gA = A'|_{B^4\setminus\{0\}}$$

(Analogous to removable singularities for holomorphic functions).

Using this, we proved the theorem. $\square$

<u>Rmk:</u> 1) The change of gauge needed for removing the singularities will in general affect the ~~gd~~ global topology of the bundle over X. This is why P' from the thm is in general not isomorphic to P

2) In fact, the measure ν will have the form

$$|F_A|^2 dvol + \sum_{i=1}^{n} n_i \, \delta_{x_i}$$

with $n_i \in \mathbb{N}$ & we get $c_2(P') = c_2(P) - \sum_{i=1}^{n} n_i$