Instanton Gauge Theory

July 8th 2021

Donaldson's diagonalization theorem

Thm (Donaldson) X^4 closed, oriented, smooth, simply conn. with neg. definite intersection form ($b_2^+(X)=0$).

Then the intersection form is standard on $H_2(X;\mathbb{Z})$, i.e. there is a $\mathbb{Z}$-basis wrt which it is given by the matrix $-\mathbb{1}$.

We prove this theorem by constructing an oriented, compact cobordism $X \longrightarrow \coprod_{\{\pm e \in H^2(X,\mathbb{Z}) \mid e^2=-1\}} \mathbb{C}P^2_e$ from the ASD moduli space $\mathcal{M}(X)$.

For rest of lecture:

Let X be a closed, or., smooth 4-mfd, $P \to X$ $SU(2)$-princ. bdle, $E := P \times_{SU(2), can.} \mathbb{C}^2$ & $k := -c_2(P)$. Let A be any conn. on P.

Define $\mathcal{A}_k := \{ A + a \mid a \in L^2_3(T^*X \otimes \operatorname{ad} P) \}$
& $\mathcal{G}_k := $ (closure of $\operatorname{Aut} P$ in $L^2_4(\operatorname{End} E)$)

$\mathcal{A}_k$ is a Hilbert mfd isom to $L^2_3(T^*X \otimes \operatorname{ad} P)$
$\mathcal{G}_k$ is a Hilbert Lie group. $\mathcal{G}_k$ acts smoothly on $\mathcal{A}_k$. Regarding the quotient, we have:

<u>Thm:</u> $\mathcal{B} := \mathcal{A}_k / \mathcal{G}_k$ is Hausdorff. Let $\varepsilon > 0$ & $S_{A,\varepsilon} := \{ A + a \in \mathcal{A}_k \mid d_A^* a = 0, \|a\|_{L^2_3} < \varepsilon \}$.

Then $\mathrm{Stab}(A) \subseteq \mathcal{G}_u$ acts smoothly on $S_{A,\varepsilon}$

& we have:

i) There is $\varepsilon > 0$ s.t. the proj. $\mathcal{A}_u \to \mathcal{B}_u$ induces a homeo

$$S_{A,\varepsilon}/\mathrm{Stab}(A) \xrightarrow{\cong} U \subseteq \mathcal{B},$$

where U is a nbhd of $[A] \in \mathcal{B}_u$

ii) $\mathcal{B}_u^* := \mathcal{A}_u^*/\mathcal{B}_u$ $\left(\mathcal{A}_u^* = (\text{irred. conn.})\right)$ has the structure of a C^∞-Banach-mfd with smooth atlas given by

$$S_{A,\varepsilon} \longrightarrow \mathcal{B}_u^*$$

(note that $\mathrm{Stab}(A) = \{\pm 1\}$ acts trivially)

iii) The orbits of reducible connections in $\mathcal{B}_U$ are isolated, the action of $Stab(A) \in \{U(1), SU(2)\}$ is free on the complement of A in $S_{A,\varepsilon}$ & the homeomorphism

$$S_{A,\varepsilon}/Stab(A) \longrightarrow U$$

is a diffeom on the complement of $[A]$.

We are interested in the space $\mathcal{M}_U$ of orbits of ASD connections in $\mathcal{B}_U$. Let $[A] \in \mathcal{M}_U$ be the orbit of an ASD conn. Then a nbhd of $[A]$ in $\mathcal{B}_U$ is homeom. to $S_{A,\varepsilon}/Stab(A)$ & the preimage of $U \cap \mathcal{M}_U$

is $S_{A,\varepsilon}$ is

$$\{A+a \mid d_A^* a = 0, \|a\|_{L^2_3} < \varepsilon, F^+_{A+a} \underset{\underset{A \text{ ASD}}{\uparrow}}{=} d_A^+ a + \tfrac{1}{2}[a \wedge a]^+\}$$

$= \phi^{-1}(0)$ where

$$\phi : N := \{A+a \mid \|a\|_{L^2_3} < \varepsilon\} \longrightarrow \begin{array}{c} \Omega^0(\mathrm{ad}P) \\ \oplus \\ \Omega^2_+(\mathrm{ad}P) \end{array},$$

$$A+a \longmapsto \begin{pmatrix} d_A^* a \\ d_A^+ a + \tfrac{1}{2}[a \wedge a]^+ \end{pmatrix}$$

Fredholm maps

<u>Def:</u> Let M, M' be connected Banach mfds. A smooth map $\phi : M \longrightarrow M'$ is <u>Fredholm</u> if the hom. of Banach spaces

$$D\phi_m : T_m M \longrightarrow T_{\phi(m)} M'$$

is Fredholm for all $m \in M$. The index of $D\phi_m$ is then independent of m & called the index of ϕ.

<u>Prop:</u> Let M, M' be Banach mfds & $\phi : M \to M'$ be a Fredholm map. Then for every $m \in M$ there are nbhds $\mathcal{O}$ of m & $\mathcal{O}'$ of $\phi(m)$ & a comm. diagram

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{\phi} & \mathcal{O}' \\ \psi \big\downarrow & & \big\downarrow \psi' \\ U_0 \oplus F & \longrightarrow & V_0 \oplus G \\ (u,f) & \longmapsto & (Lu, \alpha(u,f)) \end{array}$$

where U_0, F, V_0, G are Banach spaces, F, G are finite dim with $\dim F - \dim G = \operatorname{ind}\phi$, ψ, ψ' are diffeom. onto open nbhds of 0, L is a linear iso & α is smooth with $d\alpha|_0 = 0$.

In particular, $\phi^{-1}(0)$ is homeom. to the preimage of 0 under the smooth map

$$\tilde{\phi} : F \to G, \; f \mapsto \alpha(0, f).$$

<u>Rmk:</u> In the above statement, we can choose $F = \ker d\phi_m$ & $G = \operatorname{coker} d\phi_m$ (& L as the restr. of $d\phi_m$ to any complement. subspace U_0 of $\ker d\phi_0$).

We want to show that the map

$$\phi : M \longrightarrow \Omega^0 \oplus \Omega^2_+$$

is Fredholm. Its linearization is the operator

$$d_A^* \oplus d_A^+ : \Omega^1(\mathrm{ad}P) \longrightarrow \begin{array}{c} \Omega^0(\mathrm{ad}P) \\ \oplus \\ \Omega^2_+(\mathrm{ad}P) \end{array}$$

It is not too hard to see that this operator is elliptic, using:

<u>Lemma:</u> The sequence of diff. operators

$$(\mathrm{D}) \quad 0 \longrightarrow \left(\Omega^0(\mathrm{ad}P)\right)^2_l \xrightarrow{d_A} \left(\Omega^1(\mathrm{ad}P)\right)^2_{l-1} \xrightarrow{d_A^+} \left(\Omega^2_+\right)^2_{l-2} \longrightarrow 0$$

is an elliptic complex.

<u>Pf:</u> As A is ASD, $0 = F_A^+ = d_A^+ \circ d_A$

$\boxed{F_A = d_A \circ d_A}$, hence (D) is a complex.

Its symbol sequence at $(x,\xi) \in T^*X$ is

$$0 \to \mathrm{ad}P_x \to T_x^*X \otimes \mathrm{ad}P_x \to \Lambda^2 T_x^*X \otimes \mathrm{ad}P_x \to 0$$

$$s \longmapsto \xi \otimes s$$

$$\omega \longmapsto \xi^+ \wedge \omega$$

$\boxed{\sigma(D)(x,\xi) := [D,f] \text{ where } \xi = df}$

& it is an exercise in linear algebra to show that this sequence is exact at any $\xi \neq 0$, hence (D) is elliptic. $\square$

This Lemma implies that $d_A^* \oplus d_A^+$ is Fredholm (even elliptic) with

$$\ker(d_A^* \oplus d_A^+) = H^1((D))$$

& $\operatorname{coker}(d_A^* \oplus d_A^+) = H^2((D)) \oplus H^0((D))$.

Using the Atiyah-Singer index theorem, one can show that the index of $d_A^* \oplus d_A^+$ equals

$$8c_2(P) + 3\left(b_1(X) - b_2^+(X) + 1\right)$$

We now have:

<u>Thm (Uhlenbeck)</u>: Let X be as in Donaldson's thm. For generic metrics on X, the map

$$d_A^+ : \Omega^1(\mathrm{ad}P) \longrightarrow \Omega^2_+(\mathrm{ad}P)$$

is surjective.

If A is irreducible we already know that $H^0(\Gamma(D)) = \ker d_A = \operatorname{coker} d_A^* = 0$, so in this case $d_A^* \oplus d_A^+$ is surjective, 0 is a regular value of ϕ & thus $[A] \in \mathcal{M}_k$ has a nbhd that is a smooth submanifold of $\mathcal{B}_k^*$ of dim. $\operatorname{index}(d_A^* \oplus d_A^+)$.

What about reducibles?

We shift our point of view slightly & instead look at

$$\psi : S_{A,\varepsilon} \longrightarrow \Omega^2_+(\operatorname{ad} P),$$

$$A + a \longmapsto d_A^+ a + \frac{1}{2}[a \wedge a]^+$$

$(\pi : \mathcal{A}_k \to \mathcal{B}_k)$

Then $\pi^{-1}(\mathcal{M}_k) \cap S_{A,\varepsilon} = \psi^{-1}(0)$ is $\operatorname{Stab}(A)$-invariant & we get that a nbhd of $[A] \in \mathcal{M}_k$ is homeom. to $\psi^{-1}(0) / \operatorname{Stab}(A)$.

The linearization of Ψ is

$$d_A^+ : \ker d_A^* = T_A S_{A,\varepsilon} \longrightarrow \Omega^2_+(\mathrm{ad} P).$$

From the ellipticity of (D) we get that $\mathrm{im}\, d_A^+$ is closed & that

$$\bullet \ \ker d_A^+ \big|_{\ker d_A^*} = H^1((D))$$

$$\bullet \ \mathrm{coker}\, d_A^* \big|_{\ker d_A^*} = H^2((D))$$

are fin. dim. Hence Ψ is Fredholm of index

$$\mathrm{ind}((D)) - \dim H^0((D)).$$

By the statement on Fredholm maps, there then a smooth map $\overline{\Psi} : H^1((D)) \longrightarrow H^2((D))$ & a homeom. $\overline{\Psi}^{-1}(0) \cong \pi^{-1}(\mathcal{M}_k) \cap S_{A,\varepsilon}$.

The complex (D) is equipped with a Stab(A)-action, hence Stab(A) acts on $H^1((D))$ & $H^2((D))$ & the Kuranishi map $\overline{\Psi}$ can be chosen to be Stab(A)-equivariant.

To identify this action, we note that if P reduces to an S^1-bdle Q, then writing $L = Q \times_{S^1} \mathbb{C}$, we have

$$\mathrm{ad}P = P \times_{SU(2),Ad} su(2) = \left(Q \times_{S^1} SU(2)\right) \times_{SU(2),Ad} su(2) =$$

$$= Q \times_{Ad \circ (S^1 \hookrightarrow SU(2))} su(2) = \underline{i\mathbb{R}} \oplus L^{\otimes 2}$$

bc. writing $i\mathbb{R} \oplus \mathbb{C} = su(2) \ni u = \begin{pmatrix} it & 0 \\ 0 & -it \end{pmatrix} + \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix}$,

S^1 acts trivially on $\begin{pmatrix} it & 0 \\ 0 & it \end{pmatrix}$ & by

$$\left(\rho, \begin{pmatrix} 0 & z \\ -z & 0 \end{pmatrix}\right) \longmapsto \begin{pmatrix} 0 & \rho^2 z \\ -\rho^2 z & 0 \end{pmatrix} \text{ on}$$

the second. From this, one sees that $H^1(C(D))$, $H^2(C(D))$ have non-canonical complex str. wrt which the S^1-action is the square of the canonical circle action on $\mathbb{C}^n = H^1(C(D))$ & $\mathbb{C}^m = H^2(C(D))$ (Note $S^1 \cong S^1/\{\pm 1\}$ acts canonically on H^1, H^2).

For X simply connected, $b_+^2(X) = 0$, we can choose a metric for which $H^2(C(D)) = 0$, so in this case a nbhd of a reducible orbit in $\mathcal{M}_u$ is homeomorphic to $\mathbb{C}^n/S^1$ & diffeom. to $\mathbb{C}^n/S^1$ off the vertex.

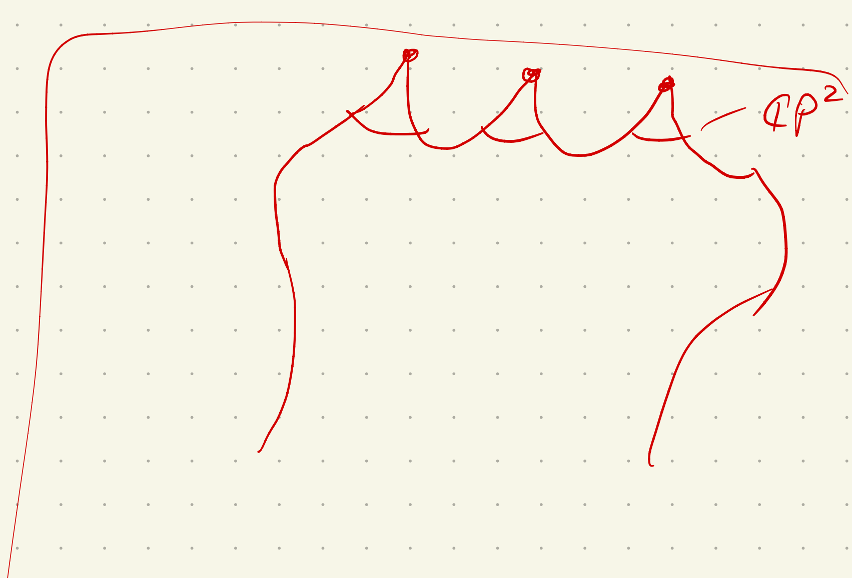
When $k=1$, the index of (D) is -5.

Hence, at a reducible (nontrivial) orbit,

bec. $H^0((D)) = \mathbb{R}$ & we can assume $H^2((D))=0$,

we get $H^1((D)) = \mathbb{R}^6 \cong \mathbb{C}^3$, so we get

<u>Thm:</u> For generic metrics on X, every reducible orbit in $\mathcal{M}_k$ has a nbhd homeom. to $\mathbb{C}^3/S^1 = \mathrm{Cone}(\mathbb{C}P^2)$. This homeom. is a diffeom. off the vertex.



What reducible orbits are there?

Prop: The orbits of the reducible conn. in $\mathcal{M}_1(X)$ for $b_1(X) = b_2^+(X) = 0$ are in 1-1-correspondence with

$$\{\pm e \in H_2(X; \mathbb{Z}) \mid e^2 = -1\}$$

The Taubes map

Thm: There is $\lambda > 0$ & smooth embedding $M \times (0, \lambda) \hookrightarrow \mathcal{M}_1$.

The image of this map is characterized as follows: An orbit $[A]$ lies in $M \times (0, \lambda)$ iff there is a point $x \in X$ & a geodesic ball B around x of radius $r < \lambda$ s.t.

$$\int_B |F_A|^2 \, dvol > 4\pi^2$$

By the characterization of $\mathrm{im}(M\times(0,\lambda)) \subseteq \mathcal{M}_1$ & the proof of Uhlenbeck compactness we get that every sequence of ASD conn. (A_i) s.t. the sequence $([A_i])$ is disjoint from $\mathrm{im}(M\times(0,\lambda))$ has a subsequence that converges up to gauge in C^∞ over all of X.

Hence we get:

Thm: $\mathcal{M}_1 \setminus M\times(0,\lambda)$ is compact. $\boxed{\Sigma}$

Now, let $[A_1],\dots,[A_n] \in \mathcal{M}_1$ be the reducible orbits & $N_1, \dots, N_n$ open nbhds homeom. to $\mathrm{Cone}(\mathbb{C}P^2)$. Then

$$\overline{\mathcal{M}} := \mathcal{M}_1 \setminus (M \times (0,\lambda) \cup N_1 \cup \dots \cup N_n)$$

is a smooth, compact 5-dim. mfd with boundary,

<u>Thm</u> (Donaldson) : $\overline{\mathcal{M}}$ is oriented.

We get a smooth, oriented, compact cobordism

$$\overline{\mathcal{M}} : X \longrightarrow \coprod_{i=1}^{n_1} \mathbb{C}P^2 \amalg \coprod_{i=1}^{n_2} \overline{\mathbb{C}P}^2$$

Because $b_2^+(X) = 0$, we

$$\mathrm{rk}\, H_2(X;\mathbb{Z}) = \mathrm{sign}(X) = \mathrm{sign}\left(\coprod^{n_1} \mathbb{C}P^2 \amalg \coprod^{n_2} \overline{\mathbb{C}P^2}\right) =$$

$= n_1 - n_2$

In particular, the number $n_1 + n_2$ of reducibles must be at least $\mathrm{rk}\, H_2(X;\mathbb{Z})$. On the other hand, there are exactly $\#\{\pm e \in H_2(X;\mathbb{Z}) \mid e^2 = -1\}$ $\leq \mathrm{rk}\, H_2(X;\mathbb{Z})$ reducible orbits in $\mathcal{M}_1$.

We thus get:

<u>Prop:</u> For X as in Donaldson's thm,

$$\mathrm{rk}\, H_2(X;\mathbb{Z}) = \#\{\pm e \in H_2(X;\mathbb{Z}) \mid e^2 = -1\}$$

It is now easy to see, using only linear algebra that this implies that the intersection form is standard

(Each $\pm e$ induces $H_2(X;\mathbb{Z}) = \langle \pm e \rangle \oplus \langle \pm e \rangle^{\perp}$)