Gauge Theory for end-periodic 4-manifolds

Thm: There is an uncountable family

$$\mathcal{T} := \{ R_s \mid 0 < s_0 \le s < \infty \}$$

of smooth 4-mfds homeom. to $\mathbb{R}^4$ but pairwise not diffeomorphic.

Let $\mathcal{R} := \{\text{oriented smooth 4-mfds homeom. to } \mathbb{R}^4\} / \text{diffeo}$

Constr. of $\mathcal{T}$:

By Freedman, there is a closed, 1-conn., oriented top. 4-mfd with intersection form

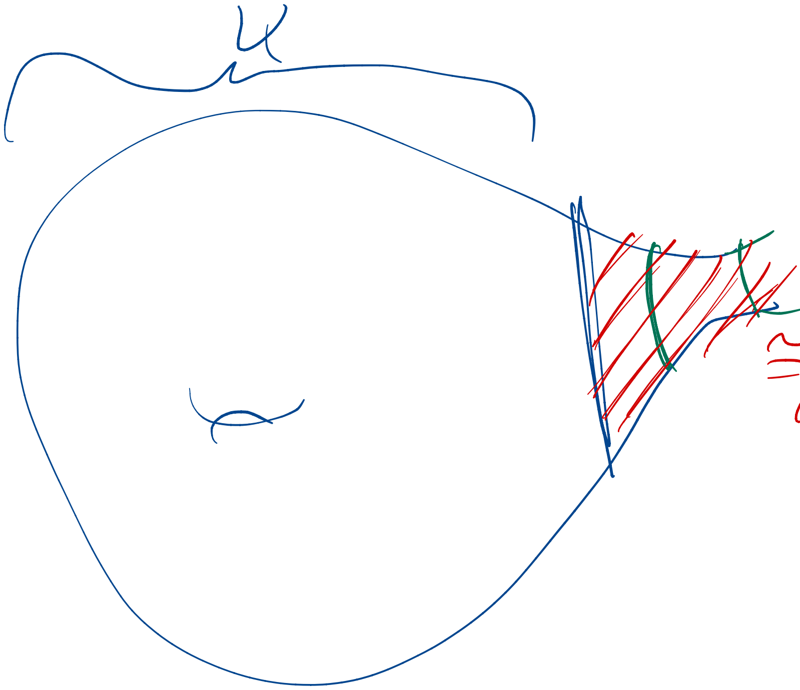
$E_8 \oplus E_8$, call this mfd $|E_8 \oplus E_8|$.

By Donaldson thm, $|E_8 \oplus E_8|$ is not smoothable.

But, by Freedman, $|E_8 \oplus E_8| \setminus pt$ is smoothable & there is $R \in \mathcal{R}$, cpt. sets $K \subseteq |E_8 \oplus E_8| \setminus pt$, $K_1 \subseteq R$ & a diffeom.

$$\left(|E_8 \oplus E_8| \setminus pt\right) \setminus K \cong R \setminus K_1 .$$

Fix a homeom. $\varphi : R \to \mathbb{R}^4$, choose $0 < s_0$ large enough that $\varphi(K_1) \subseteq B_{s_0}(0)$ & set, for $s \geq s_0$, $R_s := \varphi^{-1}(B_s(0))$ with the smooth str. induced from $R_s \hookrightarrow R$.



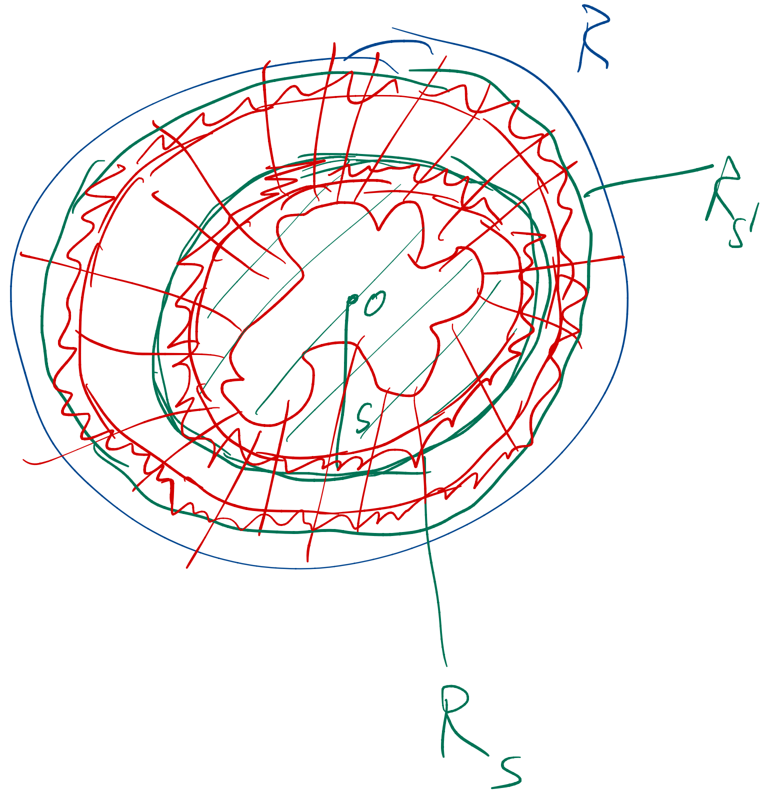
$\cong_{diff}$. (exotic $S^3 \times (0,\infty)$)

$=$

$R \setminus U_1$

$=$

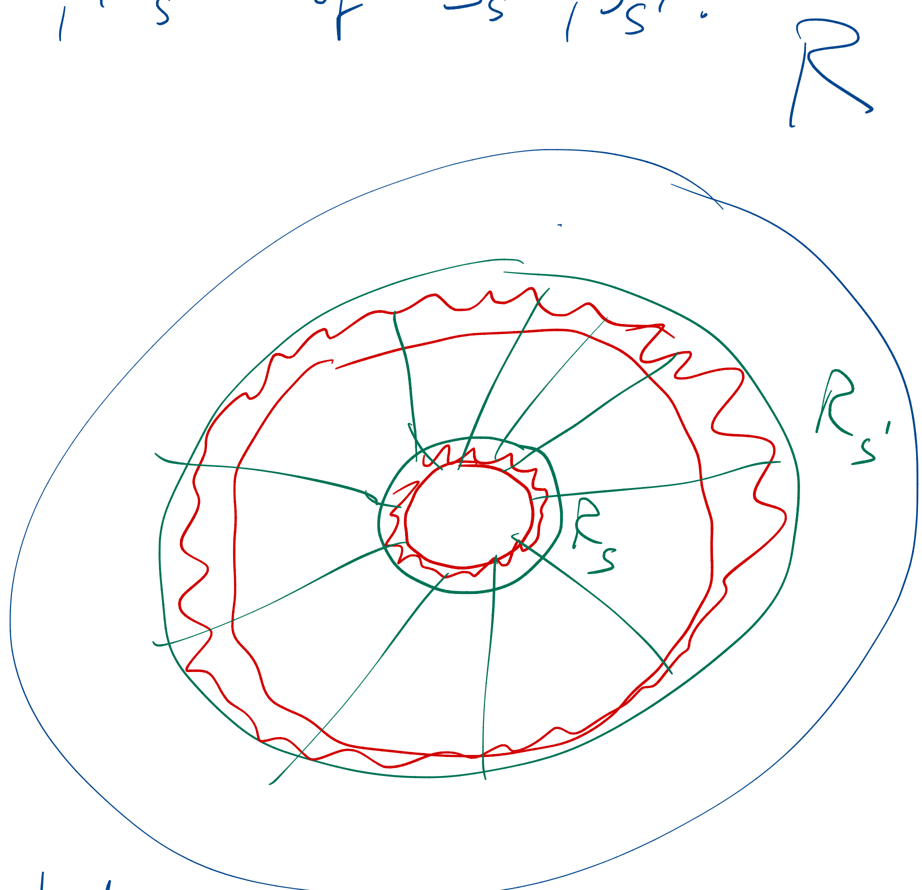
$|E_8 \oplus E_8| \setminus pt$



Assume $R_s \cong_{diff} R_{s'}$ with $s \neq s'$.

Then let S_s, $S_{s'} \subseteq R_{s+1}$ be the top. emb. bdry 3-spheres of R_s, $R_{s'}$.

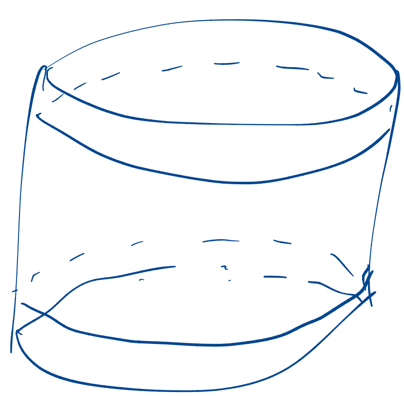
Then $R_s \cong R_{s'}$ identifies some open tubular nbhds N_s, $N_{s'}$ of S_s, $S_{s'}$.

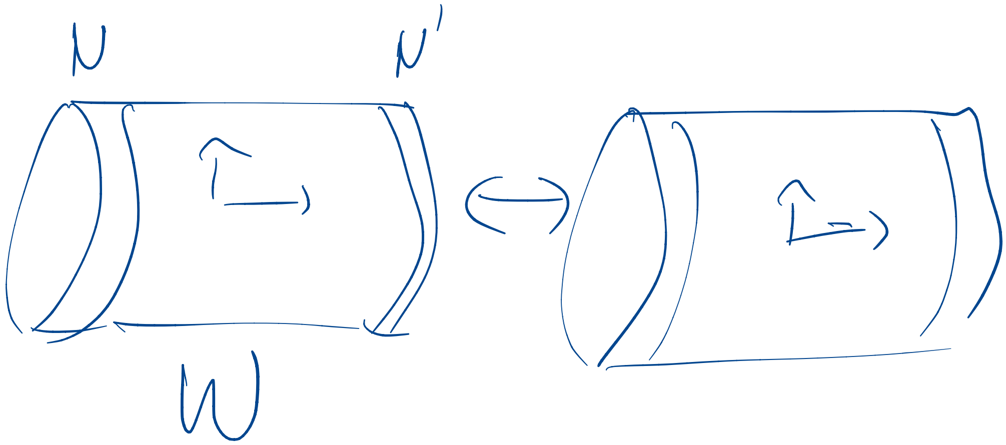


Hence

$R_{s'} \setminus (R_s \setminus N_s)$

is a smooth 4-manifold with two ends N, N' which are orientation pres. diffeom.





Using the diffeom. $|E_8 \oplus E_8| \setminus pt \cong R \setminus K_1$,

N' is diffeom. embedded in the end of $|E_8 \oplus E_8| \setminus pt$ & we can glue in

$$\underbrace{W \cup_N W \cup_N \cdots}_{\mathbb{N} \text{ copies}}$$

to get an "end periodic smoothing"

of $(E_8 \oplus E_8) \setminus pt$.

__Thm__ Let M endpos. smooth 4-mfd s.t. $\pi_1(M)$ & $\pi_1(W)$ don't have non-triv. repr. $\pi_1(M) \longrightarrow SU(2)$
$\pi_1(W) \longrightarrow SU(2)$ & $H_1(N, \mathbb{R}) = H_2(N; \mathbb{R}) = 0$
+ $b_2^-(Y) = 0$.
("admissible"), then there is a sequence of free abelian subgroups $\Lambda_{-1} \subseteq \Lambda_0 \subseteq \Lambda_1 \subseteq \dots$

s.t. $$\bigcup_{n \geq -1} \Lambda_n = H_2(M; \mathbb{Z})$$

& the induced intersection form on Λ_n is standard (i.e. isomorphic to E_n).

The endperiodic mfd above is admissible but its intersection form is $E_8 \oplus E_8$, which is not standard in the sense of the theorem.

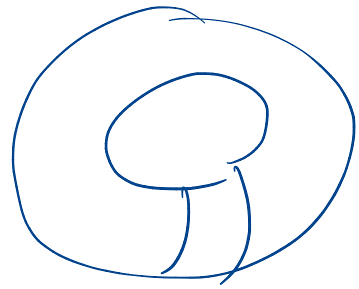
Hence, such an endperiodic smoothing does not exist.

Pf of Thm 1:

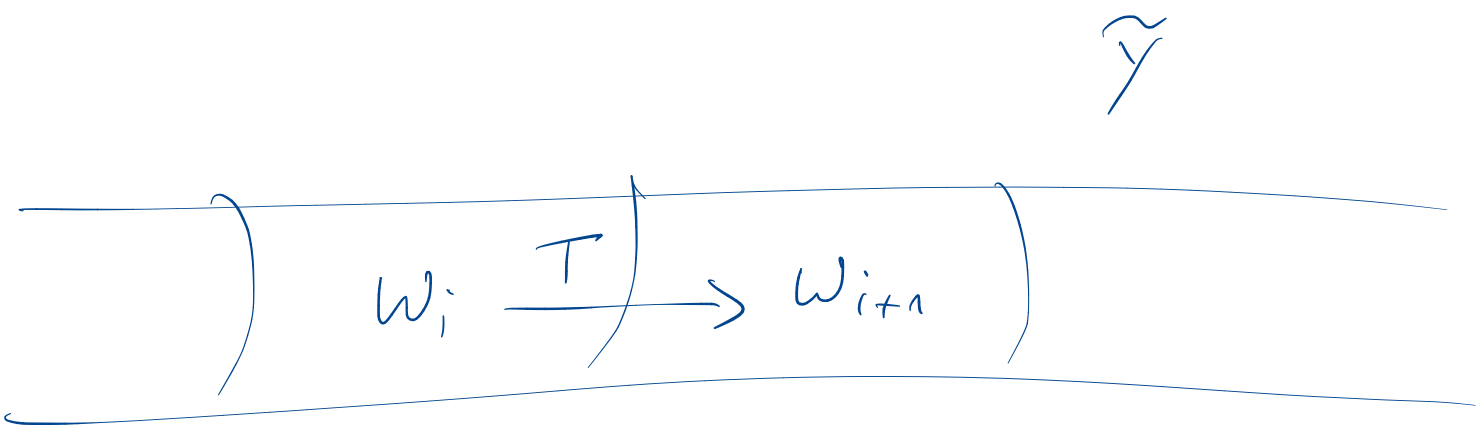
Def: A geom. str. (a bundle, conn., metric...) is endperiodic if it is isomorphic over the end of M to the restriction of the pullback of such a str.

on Y along $\pi : \tilde{Y} \to Y$

where:

$Y =$  $W/\sim \; (N \sim N')$

& $\pi : \tilde{Y} \to Y$ is the $\mathbb{Z}$-cover.



$$\mathbb{Z} = \langle T \rangle \subseteq \mathrm{Diff}(\tilde{Y})$$

$$\Rightarrow Y = \tilde{Y}/\langle T \rangle$$

Choose smooth "time function"

$$\tau : \mathrm{End}(M) \longrightarrow \mathbb{R}$$

s.t. $\tau^{-1}([n, n+1]) = W_n$ & extend to M by

$K := M \setminus \mathrm{End}M \subseteq \tau^{-1}(0)$.

Let $E \to M$ be an endperiodic vector bdle, g endperiodic metric, A endper. conn. on E, $\delta > 0$. We get Sobolev metrics on $C_0^\infty(E)$ by

$$\|s\|_{L^p_{k,\delta}} := \sum_{i=0}^{k} \left(\int_M e^{\tau\delta} \left| \nabla_A^{(k)} s \right|^p dvol_g \right)^{1/p}$$

<u>Fact</u>: Let $E|_{\mathrm{End}M} \cong M \times \mathbb{C}^2$ & A

a conn. s.t. $A|_{\text{End}\, M} = \Gamma$ (triv. prod. conn.)

Then

$$p_1(A) := \frac{1}{8\pi^2} \int_M \operatorname{tr}(F_A \wedge F_A)$$

is an integer.

Let A be as above w/ $p_1(A) = k \in \mathbb{Z}$ &

set

$$\mathcal{A}_k(\delta) := \Big\{ A_0 + a \;\Big|\; a \in L^2_{2,loc}(\operatorname{Ad} E \otimes T^*M),$$

$$(*) \; \underbrace{\int_M e^{\tau\delta} \left(|\nabla_A^{(2)} a|^2 + |\nabla_A a|^2 + |a| \right)}_{=: \|-\|_A} < \infty \Big\}$$

(Space of asympt. flat conn.)

For generic choice of δ, $\mathcal{A}_k(\delta)$ is a Banach mfd in the norm given by $(*)$.

Let $\mathcal{G}_u := \{ h \in L^2_{3,loc}(\mathrm{Aut}\, E) \,|\, \|\nabla_A h\|_A < \infty \}$.

$\mathcal{G}_u$ is a Banach Lie group. For $x \in M$,

$$r(h) := \lim_{n \to \infty} h(T^{-n}(x))$$

defines a smooth $r : \mathcal{G}_u \to G$

(where $T^n : \mathrm{End}\, M \to \{\tau \geq n\}$ is the diffeom. induced from $W_i \cong W_{i+n}$)

r is independent of x.

Up to canonical isom., $\mathcal{A}_u$ is independent of choice of A_0.

$$\mathcal{G}_u' := r^{-1}(\mathrm{id}).$$

The quotient $\mathcal{A}_u / \mathcal{G}_u' = \mathcal{B}_u'$ is a smooth Banach mfd, $SO(3) \cong \mathcal{G}_u / \mathcal{G}_u'$ acts smoothly on $\mathcal{B}_u'$ with some fixed points.

<u>Def:</u> $(E, A) \cong \left(L \oplus L^{-1}, A_1 \oplus -A_1 \right)$ as bdle w/ conn.

then this conn. is called <u>reducible</u>.

Let $\mathcal{A}_u^* := \{ a \in \mathcal{A}_u \text{ irred.} \}$.

Then $\mathcal{A}_u^* / \mathcal{G}_u$ is a smooth Banach mfd & the proj.

$$\mathcal{A}_u^* / \mathcal{G}_u' \longrightarrow \mathcal{A}_u^* / \mathcal{G}_u$$

is a smooth $SO(3)$-princ. bdle.

Choose metric g on M "asymptotically endperiodic". This defines

$$\mathcal{M}_k := \{ [a] \in \mathcal{A}_k/\mathcal{G}_k \mid F_{A_0+a} = \star_g F_{A_0+a} \}$$

<u>Thm:</u> For generic choices of δ & metric g, $\mathcal{M}_k \cap \mathcal{A}_k^*/\mathcal{G}_k$ is a smooth finite dim. mfd of dim.

$$2p_1(A_0) - 3(1 + b_1(K) - b_2^-(K)).$$

<u>Prop:</u> Assume $b_1(M) = b_2^-(M) = 0$. Then if $[A] \in \mathcal{M}_k(g)$ is the orbit of a red. conn., then a nbhd of $[A]$ is homeom. to $\mathrm{Cone}(\mathbb{CP}^2)$, this homeom. is a diffeom. off the cone point.

<u>Prop:</u> M^4 endperiodic, admissible, $b_1(K) = b_2^-(K) = 0$.

Then $\{\text{red. orbits in } \mathcal{M}_6\} \overset{1:1}{\longleftrightarrow} \{e \in H_2(M) \mid e^2 = 6\}$

If E is an $SU(2)$-bundle, then

$c_2(E) = 4 \cdot p_1(\cancel{E})$,

& we get

$\{\text{red. orbits in } \mathcal{M}_4(E)\} \overset{1:1}{\longrightarrow} \{e \in H_2(M) \mid e^2 = 1\}$.

Uhlenbeck cptness

Thm: Let $(\{A_i\})$ be a sequence in $\mathcal{M}_k(E)$, $E = SU(2)$-bdle.

i) $\exists\, x_1, \dots, x_n \in M$, a bdle $E' \to M$, a SD conn. A on E' & gauge transform. g_i s.t.

$$g_i A_i \longrightarrow A$$

in $C^\infty(M \setminus \{x_1, \dots, x_n\})$ (convergence over cpt subsets)

ii) If $p_1(A) = k$, then
$$g_i A_i \longrightarrow A \quad \text{in } C^\infty(M)$$

& if
$$\lim_{n\to\infty} \lim_{i\to\infty} \int_{t\geq n} |F_{A_i}|^2 = 0,$$

then the converse holds.

iii) If
$$\lim_{n\to\infty} \lim_{i\to\infty} \int_{t\geq n} |F_{A_i}|^2 \neq 0,$$

then $k \geq 4$, in fact

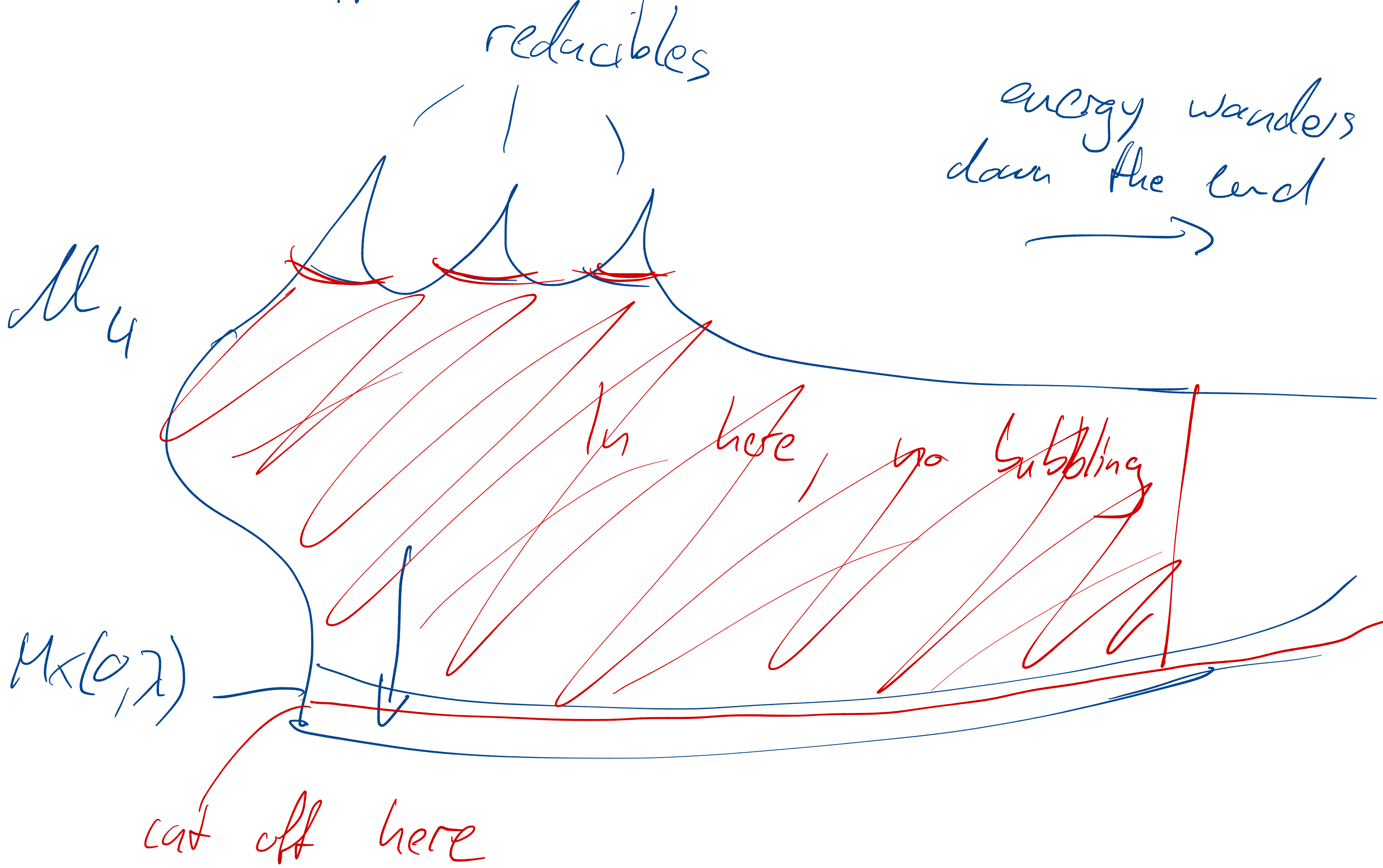
$$\lim_{n\to\infty} \lim_{i\to\infty} \int_{t\geq n} |F_{A_i}|^2 \geq 4$$

("energy can be lost over the end only in packets of 4")

There is an embedding

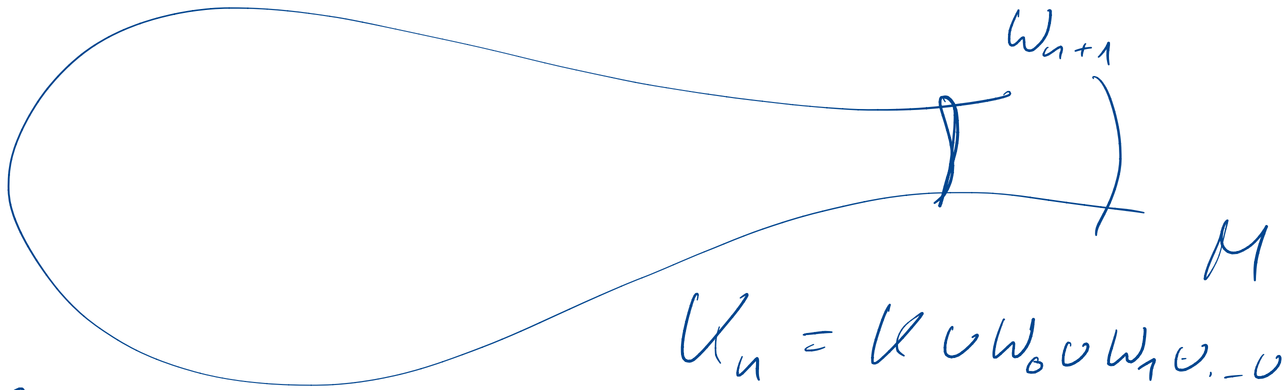
$$M \times (0, \lambda) \hookrightarrow \mathcal{M}_4(E)$$

if bubbling occurs, then the sequence will approach $M \times \{0\}$;

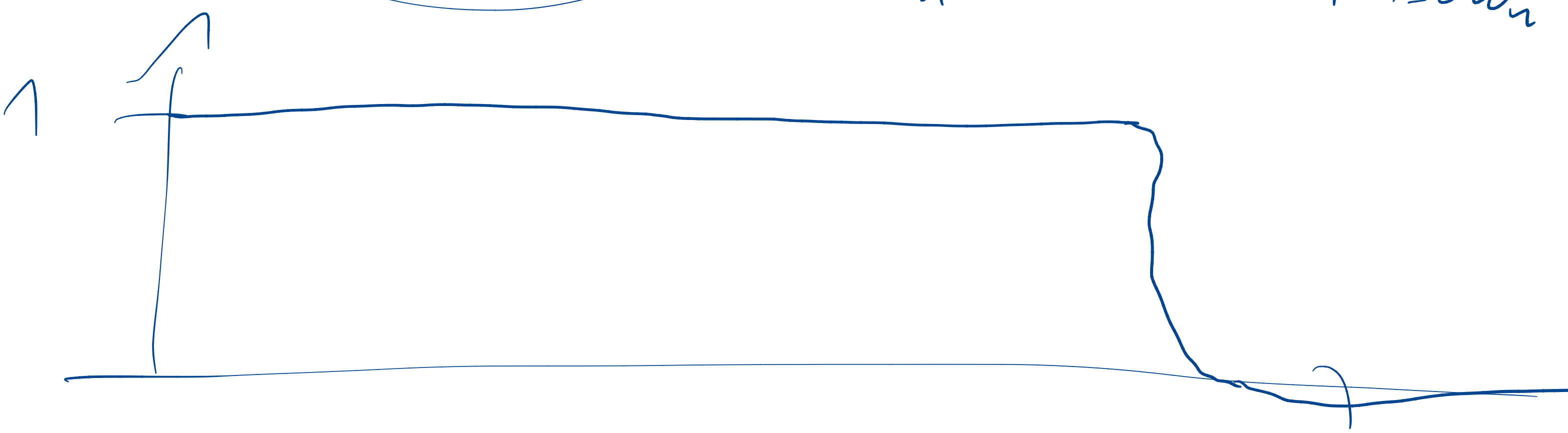


Proof of Thm 1:

Put $f([A]) := \int_M \beta \, |F_A|^2$ where



$U_n = U \cup W_0 \cup W_1 \cup \dots \cup W_n$



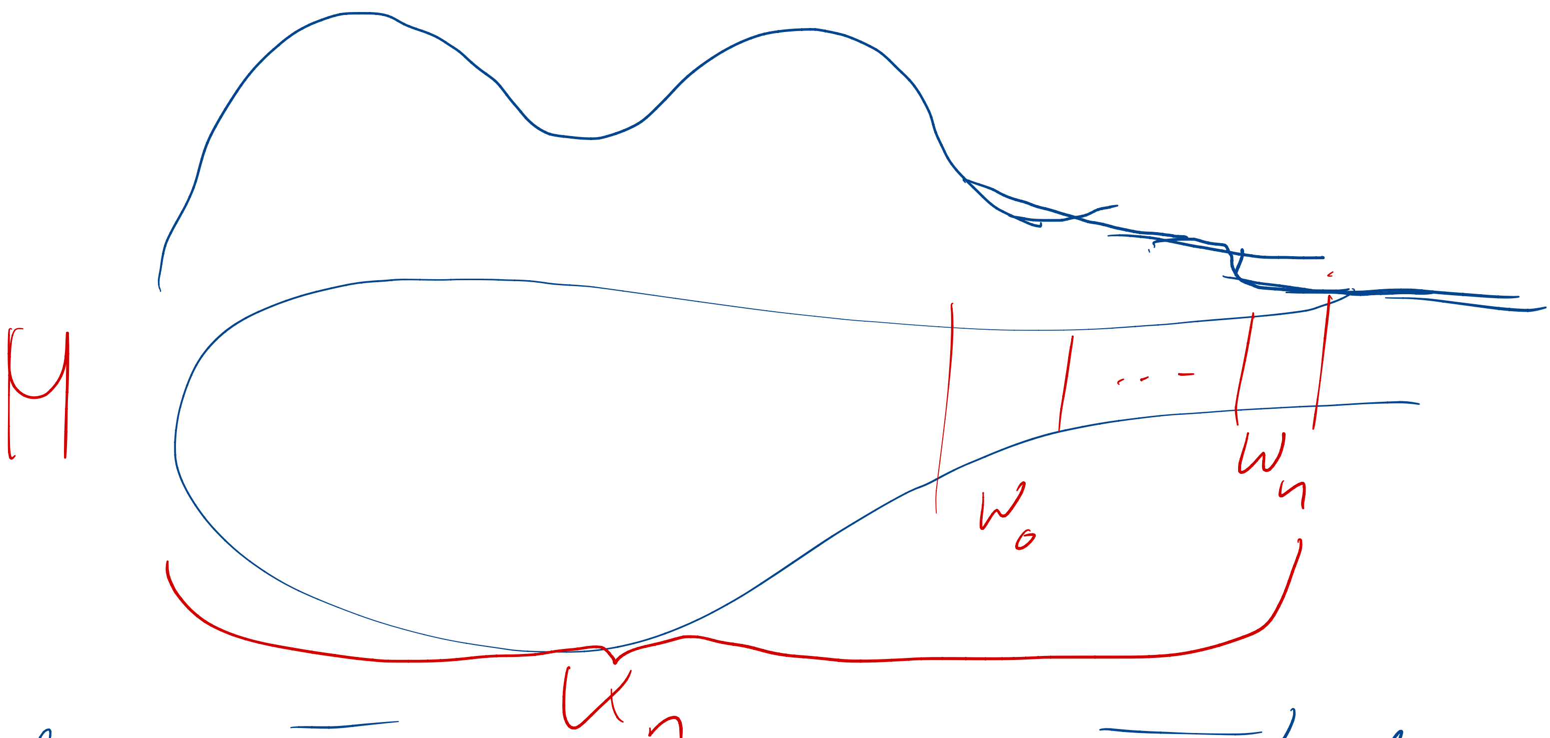
Set $\mathcal{M}^\varepsilon := \{ f \geq \varepsilon \}$ for some regular value ε.

Then $\overline{\mathcal{M}} := \mathcal{M}^\varepsilon \setminus \Big(M \times (0, \lambda_0) \cup \underbrace{\{ N_e \mid e^2 = 1 \}}_{\text{nbrhd of red.}} \Big)$

is compact.

Using β to cut of $[A] \in \overline{\mathcal{M}}$

gives an isotopy



from $\overline{\mathcal{M}}$ to a set $\overline{\mathcal{M}}'$ of

compactly supported conn's.

Let $Q_n := U_n \cup -K_n$ & E'

be $E|_{U_n} \cup (\text{triv. bundle} - K_n \times SU(2))$.

Then $\overline{\mathcal{M}}' \subseteq \mathcal{B}(E')$.

The $\mathbb{CP}^2$ - bdry components in $\overline{\mathcal{M}}'$ are in bij. with

$$\{e \in H_2(U_n; \mathbb{Z}) \mid e^2 = 1\}$$

Then repeat Donaldson's proof.

$\alpha, \beta \in H_2(U_n)$

$$\Rightarrow \alpha \cdot \beta = \sum_{e^2=1} \pm (\alpha \cdot e) \cdot (\beta \cdot e)$$

Assume $\#\{e \mid e^2 = 1\} < \operatorname{rk} H_2(K_n)$.

Then $H_2(K_n) = \langle \{e \mid e^2 = 1\} \rangle \oplus \underbrace{\langle \{e \mid e^2 = 1\} \rangle^{\perp}}_{=: U}$

For every $0 \neq \alpha \in U$

get $0 < \alpha^2 = \sum \pm \underbrace{(\alpha \cdot e)}_{=0} (\alpha \cdot e) = 0$ ↯

Thm 1 follows by setting

$$\Lambda_n := H_2(K_n)$$