Motivation: What can we do with instantons:

Donaldson's thm A:

If X^4 is a smooth oriented 4-manifold s.th.

$$q_X : H^2(X;\mathbb{Z}) \times H^2(X;\mathbb{Z}) \to \mathbb{Z}$$
$$(a,b) \mapsto \langle a \cup b, [X] \rangle$$

is negative definite. Then q_X is equivalent (over $\mathbb{Z}$) to the diagonal pairing

$$\mathbb{Z}^{b_2} \times \mathbb{Z}^{b_2} \longrightarrow \mathbb{Z}$$
$$(a,b) \mapsto a^t \begin{pmatrix} -1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{pmatrix} b$$

In contrast

<u>Freedman's thm:</u>

For any symmetric bilin. unimodular form q over $\mathbb{Z}$ $\exists$ top'l 4-manifold X (simply connected) for which

$$q_X \cong q \quad .$$

<u>Consequence:</u>

There are many top'l 4-manifolds which do not admit a smooth structure

Other results:

* Brieskorn homology 3-spheres generate a subgroup $\mathbb{Z}^\infty \subseteq \Theta^3_{\mathbb{Z}}$ (Furuta)

* Failure of the h-cob. theorem in dim = 4 (Donaldson)

* $\exists$ ~~infinitely~~ uncountably many smooth str. on $\mathbb{R}^4$ (Taubes)

  false for all $\mathbb{R}^n$, $n \neq 4$

* Property - P - conjecture (Kronheimer - Mrowka)

  $K \subseteq S^3$ knot

  $K \neq \bigcirc$ then $\exists$ irreducible

rep. $\pi_1(S^3_{p/q}(K)) \longrightarrow SO(2)$

if $|p/q| \leq 2$

* If $Y \neq S^3$ is a closed 3-mfld, then $\exists$ non-trivial rep'n

$$\pi_1(Y) \longrightarrow SL(2,\mathbb{C})$$

(Z.)

Today:

- Principal fibre bundles, associated vector bundles
- Connections

Source:

[Helga Baum: Eichfeld-theorie]

[Kobayashi-Nomizu: Foundations of Diff. geom]

1. Principal fibre bundles

<u>Def^n</u>: Let G be a Lie group. A smooth map

$\pi : P \longrightarrow M$ (P, M smooth mflds)

is called a principal fibre bundle if

* G operates freely on P from the right and is transitive on the fibres

$(pg = p \Rightarrow g = e \in G$ (freely))

$(\forall p, q \in \pi^{-1}(x)\ \exists g$ s.th. $pg = q)$

* π is locally trivial:

$\forall x \in M$ $\exists$ open nbhd $U \ni x$ and a diffeom $\Phi : \pi^{-1}(U) \to U \times G$

s.th.

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\Phi"] \arrow[d, "\pi|"'] & U \times G \arrow[dl, "pr_1"] \\ U & \end{tikzcd}$$

commutes

and Φ is G-equiv.:

If $\Phi(p) = (\pi(p), h)$, then

$$\Phi(pg) = (\pi(p), hg)$$

<u>Exercise</u>: π admits a global trivialisation iff $\pi : P \to M$ admits a section $s : M \to P$ (i.e. $\pi \circ s = \mathrm{id}_M$).

Examples:

1. Hopf bundles: $S^{2n+1} \subseteq \mathbb{C}^{n+1}$

S^1-operation:

$$((z_0, \dots, z_n), w) \mapsto (z_0 w, \dots, z_n w)$$

$$w \in S^1$$

is a free S^1-action

and

$$\pi : S^{2n+1} \longrightarrow \mathbb{C}P^n$$

$$(z_0, \dots, z_n) \longmapsto [z_0 : \dots : z_n]$$

is a principal S^1-bundle.

2. quat. Hopf bundles:

$$S^{4n+3} = \{ (q_0, \dots, q_n) \in \mathbb{H}^{n+1} \mid |q_0|^2 + \dots + |q_n|^2 = 1 \}$$

$$S^3 = \{ q \in \mathbb{H} \mid |q| = 1 \}$$

S^3 acts on S^{4n+3} in two different ways from the right:

$$S^{4n+3} \times S^3 \to S^{4n+3}$$
$$((q_0, \dots, q_n), q) \mapsto (q_0 q, \dots, q_n q)$$

$$S^{4n+3} \times S^3 \to S^{4n+3}$$
$$((q_0, \dots, q_n), q) \mapsto (\bar{q} q_0, \dots, \bar{q} q_n)$$

Then $\pi : S^{4n+3} \longrightarrow \mathbb{H}P^n$
$$(q_0, \dots, q_n) \mapsto [q_0 : \dots : q_n]$$
is a principal S^3-bundle

In particular
$$S^7 \xrightarrow{\pi} \mathbb{H}P^1 \cong S^4.$$

- Frame bundles:

If $E \xrightarrow{\pi} M$ is a

cpx
real
hermitian
euclidean

vector bundle of rank r.

$$P_E = \{(e_1,\dots,e_r) \in E\times\cdots\times E \mid (e_1,\dots,e_r) \text{ is a } \begin{matrix}\text{cpx}\\ \text{real}\\ \text{orthonormal}\end{matrix} \text{ basis of } E \text{ at the point } \pi(e_1)=\dots=\pi(e_r)\in M\}$$

has a

$$G = \begin{matrix} GL(r,\mathbb{C}) \\ GL(r,\mathbb{R}) \\ U(r) \\ O(r) \end{matrix}$$

action

$$((e_1,\dots,e_r), g) \mapsto \left(\sum_j g^{-1}_{1j} e_j, \dots, \sum_{j=1} g^{-1}_{rj} e_j\right)$$

where $g=(g_{ij})$

Alternative notations

$GL(E)$
$SL(E)$
$U(E)$
$O(E)$
$SO(E)$
$SU(E)$
$\vdots$
for P_E

This forms

$P_E \xrightarrow{\pi} M$ into a

free G-principal bundle.

- Homogeneous spaces

$H \subseteq G$ closed Lie subgroup

G/H homogen. space

Then $G \to G/H$ is a principal H-bundle

2. Associated bundles

Let $\pi : P \to M$ be a princi. G-bundle

Suppose V is a vector space and $G \xrightarrow{\rho} \mathrm{Aut}(V)$ is a group homom. Then

$P \times V$ has a right G-action

$$((p,v), g) \longmapsto (pg, \rho(g^{-1})v)$$

and

$$E := P \times_{\rho} V := P \times V / G$$

<u>Exercise:</u> $\bar{\pi} : E \to M$

$$[p,v] \longmapsto \pi(p)$$

is a vector bundle

A tautology:

E a G-vector bundle

$G = GL(n, \mathbb{K})$

$SL(n, \mathbb{K})$

$O(n)$, $U(n)$

$SO(n)$, $SU(n)$

rank $= r$

$G(E)$

$\uparrow$

G-frame bundle

Then

$$G(E) \times_G \mathbb{K}^r \longrightarrow E$$

$$[(e_1, \dots, e_r), (z_1, \dots, z_r)] \longmapsto \sum z_i e_i$$

is an isom. of vect. bundles

Further examples:

$$TM = GL(M) \times_{\rho_{can}} \mathbb{R}^n$$

$$T^*M = GL(M) \times_{\rho^*_{can}} (\mathbb{R}^n)^*$$

$$\Lambda^k M = GL(M) \times_{\rho_{can} \wedge \dots \wedge \rho_{can}} \Lambda^k(\mathbb{R}^{n*})$$

Example:

Tautological line bundle over $\mathbb{C}P^n$

$$H := \{ (\ell, \xi) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} \mid \xi \in \ell \}$$

Then

$$\begin{aligned} H &\longrightarrow L \\ (\ell, \xi) &\longmapsto \ell \end{aligned}$$

is a complex line bundle

Otoh

$$\pi : S^{2n+1} \longrightarrow \mathbb{C}P^n$$

$$\rho_k : S^1 \longrightarrow \mathrm{Aut}(\mathbb{C})$$
$$z \longmapsto (\xi \mapsto z^k \cdot \xi)$$
$$k \in \mathbb{Z}$$

<u>Exercise:</u>

$$H \cong S^{2k+1} \times_{\rho_1} \mathbb{C}$$

$$H^* \cong S^{2k+1} \times_{\rho_{-1}} \mathbb{C}$$

$$H^{\otimes k} \cong S^{2k+1} \times_{\rho_k} \mathbb{C}$$

for $k < 0$ this is $(H^*)^{\otimes |k|}$

Denote $\mathfrak{g}$ = Lie algebra of G

$$Ad : G \to Aut(G)$$
$$g \mapsto (h \mapsto ghg^{-1})$$

induces

$$ad : G \to Aut(\mathfrak{g})$$
$$g \mapsto (dAd_g)_e : \mathfrak{g} \to \mathfrak{g}$$

Diff. of Ad_g at $e \in G$

$$ad(P) := P \times_{ad} \mathfrak{g}$$

Bsp

$$(d\, ad)_e : \mathfrak{g} \to End(\mathfrak{g})$$
$$X \mapsto (Y \mapsto [X,Y])$$

Lie bracket

3. Connections in principal bundles

$$\begin{array}{c} G \longrightarrow P \\ \downarrow \pi \\ M \end{array}$$

a principal G-bdle

vertical tangent bundle:

$$VTP := \ker(d\pi : TP \to TM)$$

From the free G-action we get a linear map

$$\mathfrak{g} \longrightarrow \Gamma(TP)$$

$$\xi \longmapsto \xi^\#$$

$$\xi^\#_p := \frac{d}{dt}\Big|_{t=0} \left(p e^{t\xi}\right)$$

denotes the exponential map of G.

Observe:

$$\xi^\# \in \Gamma(VTP)$$

<u>Exercise</u>: 1. We denote by R_g the right g-action

$$\begin{tikzcd} VTP_p \arrow[d, "dR_g"'] & \mathfrak{g} \arrow[l, "\#"'] \arrow[d, "ad_{g^{-1}}"] \\ VTP_{pg} & \mathfrak{g} \arrow[l, "\#"'] \end{tikzcd}$$

commutes.

2. $$[\xi, \eta]^\# = [\xi^\#, \eta^\#]$$

Lie bracket in $\mathfrak{g}$

Lie bracket of vector fields

Observation:

$\#$ gives us a triviali-
zation

$$VTP \cong P \times \mathfrak{g}$$

Def^{n} A **connection** is a $\dim(M)$-
dimensional subbundle
$H \subseteq TP$ s.th.

1. $H \cap VTP = \{0\}$
2. $H \oplus VTP = TP$
3. $dR_g(H) = H \quad \forall g \in G.$

So a connection is a choice
of a complement of VTP
in TP.

Obs: $d\pi|_H : H \to TM$ is an isom.

Def^n If H is a connection on $P \xrightarrow{\pi} M$, then we define the associated connection 1-form

$$\omega_H \in \Omega^1(P; \mathfrak{g})$$

by the composition

$$TP_p \xrightarrow[\mathrm{pr}_{\parallel H}]{} VTP_p \xrightarrow[\cong]{(\#)^{-1}} \mathfrak{g}$$

proj. parallel to H onto VTP

__Observe:__

$$\boxed{\begin{aligned} \omega_H(X^\#) &= X \\ \text{and} \quad R_g^* \omega_H &= \mathrm{ad}_{g^{-1}} \circ \omega \end{aligned}}$$

(by previous exercise and because H is R_g-invariant)

Observe: $H = \ker(\omega_H : TP \to \mathfrak{g})$

Suppose otoh that $\omega \in \Omega^1(P;\mathfrak{g})$ satisfies

$$\omega(X^\#) = X \quad \forall X \in \mathfrak{g}$$

$$R_g^* \omega = \mathrm{ad}_{g^{-1}} \circ \omega \quad \forall g \in G.$$

Then $H_\omega := \ker(\omega : TP \to \mathfrak{g})$ is a <u>connection</u>.

<u>Observe</u>: The two constr. are inverses to each other

$$\Big(\ker \omega_H = H$$

and $$\omega_{H_\omega} = \omega \Big)$$

<u>Typical notation</u> | We write A for connection and H_A or ω_A to make explicit its manifestation.

Def^n: $\alpha \in \Omega^k(P, V)$

$\rho : G \to \mathrm{Aut}(V)$

is called

- „horizontal" if

  $\alpha(\xi_1, \dots, \xi_k) = 0$

  as soon as one of ξ_i is vertical

- „of type ρ" if

  $R_g^* \alpha = \rho(g)^{-1} \circ \alpha$

Notation:

$$\Omega^k_{\mathrm{hor}, \rho}(P; V)$$

for horiz. forms of type ρ.

Prop:

$$\Omega^k_{S,\mathrm{hori}}(P,V) \xrightarrow{\;\bar{\;}\;} \Omega^k(M; P\times_G V)$$

$$\Gamma(M; \Lambda^k T^*M \otimes P\times_G V) \quad (\text{"=" } \Omega^k(M; P\times_G V))$$

$$\omega \longmapsto \bar{\omega}$$

is an isom, where

$$\bar{\omega}_x(v_1,\dots,v_k) := [p, \omega(\xi_1,\dots,\xi_k)] \qquad v_1,\dots,v_k \in T_xM$$

equiv. class

where $\pi(p) = x$ and

$$d\pi_p(\xi_i) = v_i \quad \forall i.$$

Proof: Indep of lifts:

in $P\times_G V$: If $d\pi_p(\xi_i) = d\pi_p(\xi_i')$

$$\Rightarrow \quad \xi_i - \xi_i' \in VTP$$

ω is horiz. $\Rightarrow$

$$\omega(\dots,\xi_i,\dots) = \omega(\dots,\xi_i',\dots)$$

Indep. of $p \in \pi^{-1}(x)$:

because ω is of type S.

$\boxtimes$

Suppose ω_A and $\omega_{A'}$ are two connection 1-forms

$$\Rightarrow \omega_A - \omega_{A'} \in \Omega^1_{ad, horiz}(P, \mathfrak{g})$$

$$\Rightarrow \exists\, a \in \Omega^1_{ad, horiz}(P, \mathfrak{g})$$

s.th.

$$\omega_{A'} = \omega_A + a$$

Conclusion:

Space of connections on $P \to M$	$=$	affine space over $\Omega^1_{ad, horiz}(P, \mathfrak{g}) \cong \Omega^1(M; ad(P))$

20.4.21

Recall: $d : \Omega^k N \to \Omega^{k+1} N$:

defined by ($k>0$)

$$d\omega(\xi_0, \dots, \xi_k)$$

acts as a derivative on $\xi_0, \dots, \xi_k \in T_x N$

$$= \sum_{i=0}^{k} (-1)^i \tilde{\xi}_i \underbrace{\omega(\tilde{\xi}_0, \dots, \hat{\tilde{\xi}}_i, \dots, \tilde{\xi}_k)}$$

$$+ \sum_{i<j} (-1)^{i+j} \omega([\tilde{\xi}_i, \tilde{\xi}_j], \tilde{\xi}_0, \dots, \hat{\tilde{\xi}}_i, \dots, \hat{\tilde{\xi}}_j, \dots, \tilde{\xi}_k)$$

where $\tilde{\xi}_i$ is a vector field with $\tilde{\xi}_i(x) = \xi_i$

formula indep. of extension

$\hat{\ }$: left out

(in particular, for a 1-form

$$\boxed{d\omega(\xi, \eta) = \xi\,\omega(\eta) - \eta\,\omega(\xi) - \omega([\xi, \eta])}$$

Notice that:

d does not in general preserve $\Omega^*_{\rho,\text{horiz}}(P, V)$

Example:

$$\begin{array}{ccc} P = \mathbb{R}\times\mathbb{R} & & (t,s) \\ \downarrow & & \downarrow \\ \mathbb{R} & & t \end{array}$$

trivial $\mathbb{R}$-bdle over $\mathbb{R}$

2nd is "bundle var"

$\omega = f(s)\,dt$ is horiz.

$d\omega = \frac{\partial f}{\partial s}\, ds \wedge dt$

is not horiz.

unless $\frac{\partial f}{\partial s} \equiv 0$

If we have a connection A on the G-pr. bdle $P \to M$ with horiz. subbundle H_A, define for $\alpha \in \Omega(P, V)$

$$d_A \alpha := d\alpha \circ pr_{H_A}$$

i.e.

$$(d_A \alpha)(\xi_1, \dots, \xi_k)$$
$$= d\alpha(pr_{H_A} \xi_1, \dots, pr_{H_A} \xi_k).$$

<u>Notice</u>: $d_A \alpha$ is horizontal

(whether or not α has been)

Notice: α of type $\rho: G \to \mathrm{Aut}(V)$

$\Rightarrow d_A \alpha$ also is

(because H_A is R_g-equivariant)

In particular, get

$$d_A : \Omega^k_{\mathrm{hor},\rho}(P, V) \longrightarrow \Omega^{k+1}_{\mathrm{hor},\rho}(P, V)$$

Def$^{\mathrm{n}}$ d_A is called the covariant derivative of the connection A on P.

Preview: $d^2 = 0$, but

$d_A \circ d_A \neq 0$ in general.

Def^n: d_A descends to

$$\begin{tikzcd}
\Omega^k_{g,\mathrm{hor}}(P;V) \arrow[r, "d_A"] \arrow[d, "\cong", "-"'] & \Omega^{k+1}_{g,\mathrm{hor}}(P;V) \arrow[d, "-", "\cong"'] \\
\Omega^k(M; P_g V) \arrow[r, dashed, "\bar{d}_A"] & \Omega^{k+1}(M; P_g V)
\end{tikzcd}$$

which we denote by $\bar{d}_A$.

Def^n: Let $E \to M$ be a vector bundle. A map

$$\nabla : \Gamma(E \to M) \to \Omega^1(E \to M)$$
$$\cong \Gamma(T^*M \otimes E)$$

satisfying the Leibniz rule:

$$\nabla(fs) = df \otimes s + f \cdot \nabla s$$

$\forall f \in C^\infty(M)$, $s \in \Gamma(E)$ is called a covariant derivative

Prop^n: Let $P \to M$ be a G-principal bundle and $\delta : G \to \mathrm{Aut}(V)$, then

$$\bar{d}_A : \Gamma(M; P\times_\delta V) \longrightarrow \Omega^1(M; P\times_\delta V)$$

is a covariant derivative on the vector bundle $P\times_\delta V$.

Pf: Unravel the definitions

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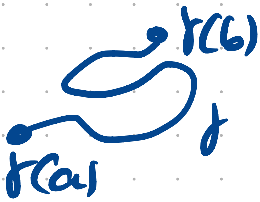
4. Parallel transport

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \big\downarrow \pi \\ & & M \end{array}$$

$\gamma : [a,b] \to M$ a smooth path.



Suppose A is a connection on P, $u \in \pi^{-1}(\gamma(a))$

Then $\exists!$ (unique) $\tilde{\gamma}_u : [a,b] \to P$

s.th. $\dot{\tilde{\gamma}}_u(t) \in (H_A)_{\gamma(t)}$

(Pf: $d\pi|_{H_A} : H_A \xrightarrow{\cong} TM$)

(Remind: H_A is a cplt of VTP in TP, which is R_g-invariant)

$$\mathrm{Par}_\gamma^A : \begin{cases} \pi^{-1}(\gamma(a)) \to \pi^{-1}(\gamma(b)) \\ u \longmapsto \tilde{\gamma}_u(b) \end{cases}$$

is called parallel transport of γ w.r.t. A.

<u>Prop:</u>
- $\mathrm{Par}_{\gamma * \mu}^A = \mathrm{Par}_\mu^A \circ \mathrm{Par}_\gamma^A$
- $\mathrm{Par}_\gamma^A \circ R_g = R_g \circ \mathrm{Par}_\gamma^A$

<u>Pf</u> Because H_A is R_g-invariant. $\square$

<u>Exercise:</u> If Par_γ^A only depends on the endpoint of γ, then the bundle $P \to M$ is trivial and A is the trivial connection ($P \cong M \times G$ $\leftarrow$ trivial conn. is $H = pr_1^* TM$)

Hint: Define a global section by parallel transport. $\boxtimes$

By Prop'n Par_γ^A descends to

$$\mathrm{Par}_\gamma^{E,A} : \begin{cases} E_{\gamma(a)} \longrightarrow E_{\gamma(b)} \\ [p, v] \longmapsto [P_\gamma^A(p), v] \end{cases}$$

on $E = P \times_G V$.

<u>Def^n</u>: Suppose ∇ is a cov. deriv. on $E \to M$.

Let $\gamma : [0,1] \to M$ be a smooth path. A section $s \in \Gamma(E \to M)$ is called <u>covariant constant</u> along γ if

$$(\nabla s)(\dot\gamma(t)) \overset{!}{=} 0 \quad \text{for all } t \in [0,1]$$

(Rk: This a dif. equ for $\gamma^* s$ on $\gamma^* E \to [0,1]$)

This also gives a notion of parallel transport

$$\mathrm{Par}_\gamma^\nabla : E_{\gamma(0)} \longrightarrow E_{\gamma(1)}$$
$$e \longmapsto s(\gamma(1))$$

if s is cov. constant along γ and $e = s(\gamma(0))$

<u>Prop</u>: If ∇_A is a cov. derivative on $E = P \times_\rho V$ from a conn. A on P, then

$$\boxed{\mathrm{Par}_\gamma^{E,A} = \mathrm{Par}_\gamma^{\nabla_A}}$$

i.e. the two notions coincide.

<u>Pf</u>: Exercise Class.

5. Curvature

Def^n Let $P \xrightarrow{\pi} M$ be a G-princip. bundle and A a connection on P. Then

$$\Omega_A := d_A \omega_A = d\omega_A \circ pr_{H_A}$$

is called the curvature of A.

Reminder: If X is an mfld and $H \subseteq TX$ is a subbundle. Then H is called <u>involutive</u> if $[\eta, \xi] \subseteq H$ for all vector fields $\eta, \xi \in \Gamma(X;H)$.

Frobenius' theorem:

Locally there are submanifolds $Y \subseteq X$ s.th. $TY = H$ iff H is involutive.

Claim: $\Omega_A \equiv 0 \iff H_A$ is involutive.

Pf: Let $\xi, \eta \in \Gamma(P; H_A)$.

$\Omega_A(\xi, \eta) = d\omega_A([\xi, \eta])$ by Def of ω_A

$= \xi . \underbrace{\omega_A(\eta)}_{\equiv 0} - \eta . \underbrace{\omega_A(\xi)}_{\equiv 0} - \omega_A([\xi, \eta])$

as derivatives

$= -\omega_A([\xi, \eta])$

$\neq 0$ iff $[\xi, \eta]$ has a vertical

component.
(because $\omega_A|_{UTP} : UTP \to \mathrm{Pr}_{\mathfrak{g}}$ is an isom.) ∎

Prop:

$$R_g^* \Omega_A = \mathrm{ad}_{g^{-1}} \Omega_A$$

Pf: $R_g^* d_A \omega_A = d\omega_A \circ pr_{H_A} \circ dR_g$

H_A is G-inv

$$= \underbrace{d\omega_A \circ dR_g}_{= R_g^* \omega_A} \circ pr_{H_A}$$

$$= d R_g^* \omega_A \circ pr_{H_A}$$

bec. $R_g^* \omega_A = \mathrm{ad}_{g^{-1}} \omega_A$

$$= d\, \mathrm{ad}_{g^{-1}} \omega_A \circ pr_{H_A}$$

commutes

$$= \mathrm{ad}_{g^{-1}} \circ \Omega_A$$

∎

So $\Omega_A \in \Omega^2_{\text{horiz}, \text{ad}}(P; \mathfrak{g})$

Under $\Omega^2_{\text{horiz}, \text{ad}}(P; \mathfrak{g}) \cong \Omega^2(M; \underset{\underset{P \times_{\text{ad}} \mathfrak{g}}{\uparrow}}{\text{ad}(P)})$

we denote the image by $F_A := \overline{\Omega_A}$

<u>Prop</u> (Cartan's formula)

$$\Omega_A = d\omega_A + \frac{1}{2}[\omega_A \wedge \omega_A]$$

(hybrid notation for $[\,,\,] \otimes \wedge$)

Pf: Check for $\Omega_A(\xi,\eta)$ ...

- ξ,η both vertical vector fields, wlog $\xi = X^\#_p$, $\eta = Y^\#_p$.

LHS $\equiv 0$

RHS

$$\left(d\omega_A + \tfrac{1}{2}[\omega_A \wedge \omega_A]\right)(X^\#, Y^\#)$$

$$= d\omega_A(X^\#, Y^\#) + \tfrac{1}{2}[\omega_A(X^\#), \omega_A(Y^\#)] - \tfrac{1}{2}[\omega_A(Y^\#), \omega_A(X^\#)]$$

$$= X^\#\underbrace{\omega_A(Y^\#)}_{=Y,\ \text{const}} - Y^\#\underbrace{\omega_A(X^\#)}_{=X} - \omega_A([X^\#, Y^\#]) \qquad \equiv 0 \qquad \equiv 0$$

$$+ [\underbrace{\omega_A(X^\#)}_{=X}, \underbrace{\omega_A(Y^\#)}_{=Y}]$$

$$= -\omega_A([X,Y]^\#) + [X,Y]$$

$$= -[X,Y] + [X,Y] = 0$$

- One horiz, one vertical $X^\#$ $\leftarrow$ $\tilde{v}$ G-inv. horiz lift of $v \in \Gamma(TM)$

LHS $= 0$

RHS:

$$d\omega_A(\tilde{v}, X^\#) = \tilde{v} \cdot \omega_A(X^\#) - X^\# \underbrace{\omega_A(\tilde{v})}_{=0} - \omega_A([\tilde{v}, X^\#])$$

$$[X^\#, \tilde{v}]_p = \frac{d}{dt}\Big|_{t=0} \underbrace{d(R_{e^{tX}}) \tilde{v}_{pe^{tX}}}_{\equiv \tilde{v}_p}$$

bec. $\tilde{v}$ is G-inv. horiz. lift

$= 0$

$= 0$

$$[\omega_A \wedge \omega_A](\tilde{v}, X^\#) = 0$$

- both hor. wlog $\tilde{v},\tilde{w}$ (Rg - inv.)

LHS:

$$\Omega_A(\tilde{v},\tilde{w}) = d\omega_A(\tilde{v},\tilde{w})$$

$$= \tilde{v}\,\underbrace{\omega_A(\tilde{w})}_{\equiv 0} - \tilde{w}\,\underbrace{\omega_A(\tilde{v})}_{\equiv 0}$$

$$- \omega_A([\tilde{v},\tilde{w}])$$

RHS:

$$\left(d\omega_A + \tfrac{1}{2}[\omega_A \wedge \omega_A]\right)(\tilde{v},\tilde{w})$$

$$= d\omega_A(\tilde{v},\tilde{w}) + 0 \quad \blacksquare$$

Prop: Let $\alpha \in \Omega^1_{S, \text{horiz}}(P; V)$

Then

$$d_A \alpha = d\alpha + \rho_*(\omega_A) \wedge \alpha$$

where

hybrid notation for $\rho_* \otimes \wedge$

$\rho: G \to \mathrm{Aut}(V)$

$\rho_*: \mathfrak{g} \to \mathrm{End}(V)$ its derivative

Pf: Just as with Cartan's formula on

- vertical / horiz
- horiz / horiz
- vert / vert

Rk: Also true for $\alpha \in \Omega^k_{S, \text{horiz}}(P; V)$

$$\left(\rho_*(\omega_A) \wedge_k \alpha\right)(\xi_0, \ldots, \xi_k)$$

$$= \sum_{i=0} (-1)^i \rho_*(\omega_A(\xi_i)) \alpha(\xi_0, \hat{\xi}_i, \ldots \xi_k)$$

– –

Recall for any two conn. A', A
$\exists$ 1-form $a \in \underset{\substack{S\text{-eq.}\\ \text{conv.}}}{\Omega^1(P; \mathfrak{g})}$ s.th.

$$\omega_{A'} = \omega_A + a.$$

Prop^n: $\Omega_{A+a} = \Omega_A + d_A a + \frac{1}{2}[a \wedge a]$

Pf: $\Omega_{A+a} \overset{\text{Cartan form.}}{=} d\omega_{A+a} + \frac{1}{2}[\omega_{A+a} \wedge \omega_{A+a}]$

$$= d\omega_A + da + \frac{1}{2}[\omega_A \wedge \omega_A]$$

Rk:
For 1-forms:
$\alpha \wedge \beta$
$= (-1) \cdot \beta \wedge \alpha$

$$+ \frac{1}{2}[\omega_A \wedge a] + \underbrace{\frac{1}{2}[a \wedge \omega_A]}_{= \frac{1}{2}[\omega_A \wedge a]}$$

$$+ \frac{1}{2}[a \wedge a]$$

because of Lie bracket

$$= \Omega_A + da$$

$$+ [\omega_A \wedge a] + \frac{1}{2}[a \wedge a]$$

$$\underset{\substack{\text{Prop'n}\\ \text{applied with}\\ \rho = \mathrm{ad} \Rightarrow \rho_* = [-,\ ]}}{\overset{\text{previous}}{=}} \Omega_A + d_A a + \frac{1}{2}[a \wedge a]$$

$\blacksquare$

Prop^n (Bianchi identity)

$$d_A \Omega_A = 0$$

Pf: $d_A \Omega_A (\xi, \eta, \lambda)$

$$\overset{\text{Prop}^n}{=} d\Omega_A(\xi,\eta,\lambda) + [\omega_A \wedge \Omega_A](\xi,\eta,\lambda)$$

$$\overset{\text{Cartan's formula}}{=} \frac{1}{2} d[\omega_A \wedge \omega_A](\xi,\eta,\lambda) + [\omega_A \wedge d\omega_A](\xi,\eta,\lambda) + \left[\omega_A \wedge \frac{1}{2}[\omega_A \wedge \omega_A]\right](\xi,\eta,\lambda)$$

changes sign

$$= \left.\begin{aligned} &\frac{1}{2}[d\omega_A \wedge \omega_A](\dots) \\ &- \frac{1}{2}[\omega_A \wedge d\omega_A](\dots) \\ &+ [\omega_A \wedge d\omega_A](\dots) \end{aligned}\right] = 0$$

$$+ \frac{1}{2}[\omega_A \wedge [\omega_A \wedge \omega_A]](\dots)$$

$$= \frac{1}{2} [\omega_A \wedge [\omega_A \wedge \omega_A]] (\dots)$$

wlog $\xi, \eta, \lambda = X^\#, Y^\#, Z^\#$

for $X, Y, Z \in \mathfrak{g}$

$= 0$ because of

Jacobi's identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

$\blacksquare$

Prop: For $\alpha \in \Omega^k_{hor,\rho}(P,V)$ and A a connection on P we have

$$\boxed{d_A d_A \alpha = \rho_*(\Omega_A) \wedge \alpha}$$

Proof:

$$d_A d_A \alpha \overset{\text{Prop in}}{\underset{\text{above}}{=}} d(d\alpha + \rho_*(\omega_A)\wedge\alpha) + \rho_*(\omega_A) \wedge (d\alpha + \rho_*(\omega_A)\wedge\alpha)$$

$$= \rho_*(d\omega_A)\wedge\alpha - \cancel{\rho_*(\omega_A)\wedge d\alpha} + \cancel{\rho_*(\omega_A)\wedge d\alpha} + \rho_*(\omega_A)\wedge\rho_*(\omega_A)\wedge\alpha$$

Now

$$(\rho_*(\omega_A)\wedge\rho_*(\omega_A))(\xi,\eta)$$

$$=(\rho_*(\omega_A)(\xi))\,\rho_*(\omega_A(\eta))$$
$$\quad-\rho_*(\omega_A(\eta))\,\rho_*(\omega_A(\xi))$$

$$=[\rho_*(\omega_A(\xi)),\ \rho_*(\omega_A(\eta))]$$

the bracket in End(V)

$$=\rho_*([\omega_A(\xi),\omega_A(\eta)])$$

ρ_* Lie alg. homom.

$$=\rho_*\left(\tfrac{1}{2}[\omega_A\wedge\omega_A]\right)(\xi,\eta)$$

$\Longrightarrow$ Result follows from Cartan's formula

Last step:

$$\omega = \sum \overset{\in \mathfrak{g}}{X_i} \; \underset{\Omega^1(P)}{\underset{\cap}{\alpha_i}}$$

$$[\omega_A \wedge \omega_A](\xi, \eta)$$

$$= \sum_{i,j} [X_i, X_j] \; \underbrace{\alpha_i \wedge \alpha_j (\xi, \eta)}$$

$$= \alpha_i(\xi)\, \alpha_j(\eta) - \alpha_i(\eta)\, \alpha_j(\xi)$$

$$= [\omega_A(\xi), \omega_A(\eta)] - [\omega_A(\eta), \omega_A(\xi)]$$

$$= 2\, [\omega_A(\xi), \omega_A(\eta)] \qquad \blacksquare$$

The <u>curvature</u> of a cov. derivative

$$\nabla : \Gamma(E) \to \Gamma(\Lambda^1 T^*M \otimes E)$$

is defined by

$$R^{\nabla}(\xi,\eta) = \nabla_\xi \nabla_\eta - \nabla_\eta \nabla_\xi - \nabla_{[\xi,\eta]}$$

vf. on M

Prop: If A is a conn on $\begin{matrix} P \\ \downarrow \\ M \end{matrix}$

On $P \times_G V$ we had got a covariant derivative ∇^A induced from A.

Then

$$R^{\nabla^A} = \rho_*(F_A) \qquad P\times_{\mathrm{ad}}\mathfrak{g}$$

where $F_A \in \Omega^2(M; \mathrm{ad}(P))$

$$\rho_* : \mathfrak{g} \to \mathrm{End}(V)$$

Pf: Exercise !

$\blacksquare$

Next time:

$$\begin{tikzcd} G \curvearrowright P \arrow[rr, dashed, "f"] \arrow[dr] & & Q \curvearrowleft H \arrow[dl] \\ & M & \end{tikzcd}$$

bundle hom

H

$\varphi: G \to H$ is a Lie group hom.

f is called a bundle map if $f(pg) = f(p)\varphi(g)$ $\forall p, g$.