Chern-Weil theory I

$\Phi : \mathfrak{g} \longrightarrow \mathbb{R}$ (or $\mathbb{C}$)

Lie alg of G

polynomial of deg $= k$

alternatively

$\Phi : \mathfrak{g} \times \dots \times \mathfrak{g} \longrightarrow \mathbb{R}$

multilin.
invariant under permutations
("symmetric")

suppose ad-invariant:

$\Phi(\mathrm{ad}_g X_1, \dots, \mathrm{ad}_g X_k)$

$= \Phi(X_1, \dots, X_k)$

$\forall g \in G$

$\forall X_1, \dots, X_k \in \mathfrak{g}$

Apply this to $g = e^{tX}$
& diff'te at $t=0$

$$\Rightarrow \quad (*) \left[\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0} \Phi(\mathrm{ad}_{e^{tX}} X_1, \dots, \mathrm{ad}_{e^{tX}} X_k) \\ &= \Phi([X, X_1], X_2, \dots, X_k) \\ &\quad + \Phi(X_1, [X, X_2], \dots, X_k) \\ &\quad \dots\dots \end{aligned} \right.$$

Let A be a connection $\begin{matrix} P \\ \downarrow \\ M \end{matrix}$

$$c_\Phi(A) := \Phi(\underbrace{\Omega_A \wedge \dots \wedge \Omega_A}_{k \text{ times}})$$

$$\in \Omega^{2k}_{hor}(P)$$

Ω_A : curvature of A

Two facts from last time:

- $d_A \Omega_A = 0$ Bianchi id.
- $\alpha \in \Omega_{horiz,\rho}(P; V)$ $\rho: G \to \mathrm{Aut}\, V$

then

$$(**) \quad d_A \alpha = d\alpha + \underbrace{\rho_*(\omega_A) \wedge \alpha}_{\text{legbraid contraction}}$$

$(ad)_* = [\ ,-]$

$\rho_*: \mathfrak{g} \to \mathrm{End}\, V$

<u>Prop^n</u>:

$c_\phi(A)$ is closed, and for any other connection A' on P the difference

$$c_\phi(A) - c_\phi(A') \text{ is } \underline{\text{exact}}.$$

Proof:

$$dc_\phi(A) = \phi(d\Omega_A \wedge \Omega_A \wedge \Omega_A \cdots) + \phi(\Omega_A \wedge d\Omega_A \wedge \Omega_A \cdots)$$

$$= k \cdot \phi(d\Omega_A \wedge \Omega_A \wedge \Omega_A \cdots)$$

$$\overset{(*)}{=} k \cdot \phi((d\Omega_A + [\omega_A \wedge \Omega_A]) \wedge \Omega_A \wedge \cdots \wedge \Omega_A)$$

In fact $(*)$ implies:

$$0 = \phi([\omega_A \wedge \Omega_A] \wedge \Omega_A \wedge \cdots \wedge \Omega_A) + \phi(\Omega_A \wedge [\omega_A \wedge \Omega_A] \wedge \cdots \wedge \Omega_A) + \cdots$$

$$= k \cdot \phi(\underbrace{d_A \Omega_A}_{=0 \text{ by Bianchi id.}} \wedge \Omega_A \wedge \cdots)$$

$$= 0$$

Let A' be another conn.

$$a := A' - A \in \Omega^1_{\text{hor}, \text{ad}}(P; \mathfrak{g})$$

$A_t = A + ta$ is a path of conn. from A to A'.

Then

$$\Omega_{A_t} = \Omega_A + d_A(ta) + \frac{1}{2} t^2 [a \wedge a]$$

$$\Rightarrow \frac{d}{dt} \Omega_{A_t} = d_A a + t [a \wedge a]$$

$$= d_{A_t} a$$

$$\frac{d}{dt} c_\Phi(A_t)$$

$$= k \cdot \Phi\left(\frac{d\Omega_{A_t}}{dt} \wedge \Omega_{A_t} \wedge \dots \wedge \Omega_{A_t}\right)$$

$$= k \cdot \Phi\left(d_{A_t} a \wedge \Omega_{A_t} \wedge \dots \wedge \Omega_{A_t}\right)$$

<u>Lemma:</u>

If $\beta \in \Omega^*_{\text{horiz}}(P;\mathbb{R})$ (trivial G-action) G-invariant,

then

$$d_A \beta = d\omega$$

<u>Pf:</u> Use $(**)$, with δ: trivial rep'n $\boxed{=}$

$$\underset{\text{Bianchi identity}}{=} k \cdot d_{A_t} \Phi(a \wedge \Omega_{A_t} \wedge \dots \wedge \Omega_{A_t})$$

$$\underset{\text{Lemma}}{=} k \cdot d\Phi(a \wedge \Omega_{A_t} \wedge \dots \wedge \Omega_{A_t})$$

Integration

$$\Longrightarrow \quad c_\phi(A') - c_\phi(A) = d\left(k \cdot \int_0^1 \phi(a \wedge \Omega_{A_t} \wedge \dots \wedge \Omega_{A_t})\right)$$

$\blacksquare$

Example:

If $\mathfrak{g}$ matrix Lie alg. of matrix Lie group G, then

$$\det(t \cdot \mathrm{Id} + X) =: \sum_{k=0} t^k \phi_k(X)$$

ad_G-inv. polynomial of degree (?) $\mathrm{rk}(G) - k$

Example: $G = U(1)$

$$\mathfrak{g} := \mathfrak{u}(1) = i\mathbb{R}$$

Remark: If $P = M \times G$ trivial bundle, then it admits the trivial connection $pr_1^* TM$, which has 0 curvature (is integrable).

$$\Rightarrow [c_\Phi(\text{triv. conn})] = 0$$

Lemma:

If $\rho : G \to Aut(V)$ is the trivial hom., then

$$\Omega^*(M; \underbrace{P \times_\rho V}_{\substack{= M \times V \\ \text{triv. bdle}}}) \xrightarrow{\cong} \Omega^*_{\substack{\text{basic} \\ \rho\text{-equiv}}}(P; V)$$

this is given by π^* $\square$

Notice that

$$d\pi^* = \pi^* d$$

Therefore $\exists!$ class

$$\bar{c}_\phi(A) \in \Omega(M;\mathbb{C})$$

s.t. $\quad \pi^*\bar{c}_\phi(A) = c_\phi(A).$

In fact

$$\bar{c}_\phi(A) = \phi(F_A \wedge \cdots \wedge F_A)$$

where

$$F_A \in \Omega^2(M;\mathrm{ad}(P))$$

vector bundle with fibre $\mathfrak{g}$

We have

$$\bar{c}_\phi(A) - \bar{c}_\phi(A') \in d\,\Omega^1(M;\mathbb{C})$$

if A and A' are two conn.

$$\Rightarrow [\bar{c}_\phi(A)] \in H^{2k}_{dR}(M;\mathbb{C})$$ only depends on P!

Example:

$$\begin{tikzcd} S^1 \arrow[r] & S^3 \arrow[d] & \subseteq \mathbb{C}^2 \\ & \mathbb{CP}^1 \cong S^2 & \end{tikzcd}$$

$$(z,w) \mapsto [z:w]$$

Hopf fibration

We will apply the above to

$$c_\phi(A) = -\frac{1}{2\pi i} \Omega_A \in \Omega^2(S^3;\mathbb{R})$$

for some conn. A

- ad-action is trivial for $G = S^1$

What is $[\overline{c}_\phi(A)] \in H^2_{dR}(S^2;\mathbb{R})$?

de Rham isomorphism:

$$H^2_{dR}(S^2) \longrightarrow \mathbb{R}$$
$$[\omega] \longmapsto \int_{S^2} \omega$$

Chart for $\mathbb{CP}^1$ is

$$\mathbb{C} \xrightarrow{\Phi} \mathbb{CP}^1$$
$$u \longmapsto [u:1]$$

Then $\Phi(\mathbb{C}) = \mathbb{CP}^1 \setminus \{[1:0]\}$

Exercises The:

$$\omega_A = \bar{w}\,dw + \bar{z}\,dz$$

$$\Omega_A = d\omega_A$$

(no quadratic term because $G = S^1$)

$$= d\bar{w} \wedge dw + d\bar{z} \wedge dz$$
$$= -\,dw \wedge d\bar{w} - dz \wedge d\bar{z}$$

Need to find $F_A \in \Omega^2(S^2; i\mathbb{R})$

s.th. $\pi^* F_A = \Omega_A$

We will express F_A through the dual Φ

In fact:

$$F_A\big|_{\Phi(\mathbb{C})} = s^* \Omega_A$$

because then

$$\pi^* F_A = \pi^* s^* \Omega_A$$
$$= \Omega_A$$

bec. Ω_A is S^1-invariant.

$$\begin{tikzcd} \pi^{-1}(\Phi(\mathbb{C})) \subset S^3 \arrow[d, shift left, "\pi"] \\ \Phi(\mathbb{C}) \arrow[u, bend left, "s"] \end{tikzcd}$$

Looking for a section

$\pi \circ s = id$

$\Rightarrow s^* \circ \pi^* = id$

$$
\begin{tikzcd}
\Phi^*(S^3 \to S^2) \arrow[r, dashed] \arrow[d] & \pi^{-1}(\Phi(\mathbb{C})) \subseteq S^3 \arrow[d] \\
\mathbb{C} \arrow[r, "\Phi"'] & \Phi(\mathbb{C}) \subseteq S^2 \arrow[u, bend left, "s"]
\end{tikzcd}
$$

A candidate is

$$s([u:1]) = \frac{(u,1)}{\sqrt{|u|^2+1}}$$

(defined as

$$(**) \quad u \longmapsto \frac{(u,1)}{\sqrt{|u|^2+1}}$$

composed with Φ^{-1})

$$F_A = s^* \Omega_A$$

$$\Rightarrow \Phi^* F_A = \Phi^* s^* \Omega_A$$

$$= (s \circ \Phi)^* \Omega_A$$

$$(**) = s \circ \Phi$$

$$\Omega_A = -\left(dz \wedge d\bar{z} + dw \wedge d\bar{w} \right)$$

$$\Rightarrow (s \circ \phi)^* \Omega_A$$

$$= -\left(d\left(\frac{u}{\sqrt{|u|^2+1}}\right) \wedge d\left(\frac{\bar{u}}{\sqrt{|u|^2+1}}\right) + 0 \right)$$

$$d\left(\frac{u}{\sqrt{|u|^2+1}}\right) = \underbrace{\frac{du}{\sqrt{|u|^2+1}}}_{1} - \frac{1}{2} u \underbrace{\frac{d|u|^2}{(|u|^2+1)^{3/2}}}_{2}$$

$$= \frac{du}{\sqrt{|u|^2+1}} - \frac{1}{2} u \frac{du \cdot \bar{u} + u\, d\bar{u}}{(|u|^2+1)^{3/2}}$$

$$d\left(\frac{\bar{u}}{\sqrt{|u|^2+1}}\right) = \frac{d\bar{u}}{\sqrt{|u|^2+1}} - \frac{1}{2} \bar{u} \left(\frac{du \cdot \bar{u} + u\, d\bar{u}}{(|u|^2+1)^{3/2}}\right)$$

$$= -\left(\underbrace{\frac{du \wedge d\bar{u}}{|u|^2+1}}_{1 \wedge 1} - \underbrace{\frac{1}{2} \frac{|u|^2}{(|u|^2+1)^2} du \wedge d\bar{u}}_{1 \wedge 2} \right.$$

$$\left. - \underbrace{\frac{1}{2} \frac{|u|^2}{(|u|^2+1)^2} du \wedge d\bar{u}}_{2 \wedge 1} \right.$$

$$+\frac{1}{4}\frac{|u|^4}{(|u|^2+1)^3}\,du\wedge d\bar u$$

$$+\frac{1}{4}\frac{u^2\bar u^2}{(|u|^2+1)^3}\,d\bar u\wedge du\Big)$$

$=0$

$$=-\frac{du\wedge d\bar u}{(1+|u|^2)^2}$$

$$(dx+i\,dy)$$
$$\wedge(dx-i\,dy)$$
$$=-2i\,(dx\wedge dy)$$

$$\Rightarrow\int_{\mathbb{CP}^1}\Big(-\frac{1}{2\pi i}F_A\Big)=\int_{\phi(\mathbb{C})}\Big(-\frac{1}{2\pi i}\Big)F_A$$

$$=\int_{\mathbb{C}}-\frac{1}{2\pi i}\,\phi^*F_A$$

$$=\int_{\mathbb{C}}\frac{1}{2\pi i}\,\frac{du\wedge d\bar u}{(1+|u|^2)^2}$$

$$= -\int_{\mathbb{C}} \frac{1}{\pi} \frac{dx \wedge dy}{(1+|w|^2)^2}$$

$$= -\frac{1}{\pi} \int_0^{2\pi} \left(\int_0^{\infty} \frac{r\,dr}{(1+r^2)^2} \right) d\varphi$$

$$= -2 \cdot \int_0^{\infty} \frac{r}{(1+r^2)^2} \, dr$$

$$= -2 \cdot \left(\frac{1}{2} \left(\frac{-1}{1+r^2} \right) \right) \Big|_0^{\infty}$$

$$= -1$$

<u>Conclusion:</u>

$$-1 = \left[c(\text{Hopf-bundle}) \right] \in H^2_{dR}(\mathbb{CP}^1)$$

Reduction & Extension of structure group

Def Let $\lambda : H \to G$ be a Lie group homom.
$\pi : P \to M$ a principal G-bundle

A λ-reduction of P is a principal H-bundle $\pi' : Q \to M$ together with a map $f : Q \to P$ satisfying:

$$\begin{tikzcd} Q \arrow[rr, "f"] \arrow[dr] & & P \arrow[dl] \\ & M & \end{tikzcd}$$

$*$ commutative!

$$* \quad f(ph) = f(p)\lambda(h) \quad \forall p \in Q,\ \forall h \in H$$

$\boxed{\text{, of type } \lambda\text{"}}$

Example: $\dim M = n$

$$SO(M) \hookrightarrow GL(M)$$

oriented orthon. frame bundles

is a $SO(n)$-reduction of the frame bundle of M

(exists if TM is orientable)

Rk: P admits a λ-reduction iff $\exists$ cocycles (g_{ik}) coming from cocycles

$$h_{ik} : U_i \cap U_k \longrightarrow H$$

s.th.

$$g_{ik} = \lambda \circ h_{ik}$$

Example: $\lambda : S^1 \longrightarrow S^1$

$$z \longmapsto z^2$$

Claim: The Hopf bundle $S^3 \downarrow S^2$ does not admit a λ-reduction.

Exercise ! (Use Chern classes later on)

* A $U(n)$-principal bundle $P \to M$ admits a reduction to a $SU(n)$-principal bundle iff $P \times_{\det} \mathbb{C}$ is the trivial bundle

Btw: $\boxed{\text{Outlook}}$

· $Spin(n) \xrightarrow{2:1} SO(n)$ (unique double cover)

· Then a $SO(n)$-bundle admits a reduction to a $Spin(n)$-bundle if

$$w_2(P \times_{can} \mathbb{R}^n) = 0$$

Def: In the above situation P is called a λ-extension of Q.

Extensions always exist:

$$\begin{array}{c} Q \\ \downarrow \\ M \end{array} \qquad \begin{array}{l} H\text{-principal bdle} \\ \lambda : H \to G \end{array}$$

$$P := Q \times G /_H$$

where H acts as

$$(h, (q, g)) \mapsto (qh, \lambda(h)g)$$

Right G-action on P is induced from the right action of G on itself

$$((q, g), g') \mapsto (q, gg')$$

$Q \xrightarrow{f} P$ of type λ

is given by

$$f(q) = [q, e]$$

Reduction/extension and connections

Prop Let
$$\begin{array}{ccc} Q & \xrightarrow{f} & P \\ & {\scriptstyle \pi_Q}\searrow \quad \swarrow{\scriptstyle \pi_P} & \\ & M & \end{array}$$
and f of type $\lambda : H \to G$.
Let A be a connection on Q. Then $\exists$ unique connection A' on P s.th.
$$df_q((H_A)_q) = (H_{A'})_{f(q)} \subseteq TP \quad \forall q$$
This satisfies:
$$f^*\omega_{A'} = \lambda_* \circ \omega_A$$
$$f^*\Omega_{A'} = \lambda_* \circ \Omega_A$$
(here $\lambda_* : \mathfrak{h} \to \mathfrak{g}$ associated Lie alg. homom.)

<u>Proof:</u> Let $p = f(q)g$

$$(H_{A'})_p := dR_g\left(df_q\left((H_A)_q\right)\right)$$

<u>Check:</u> This is indep of q with the prop $\pi_P(p) = \pi_Q(q)$

(uses H-invariance of H_A and the fact that f is of type λ)

The $(H_{A'})$ then are clearly G-invariant, and they form a complement:

$$\pi_P \circ f = \pi_Q$$

$$\Rightarrow (d\pi_P)_{f(q)} \circ df_q \Big|_{H_A} : (H_A)_q \xrightarrow{\cong} TM_{\pi_Q(q)}$$

So $df_q((H_A)_q)$ is a complement to $VTP_{f(q)} = \ker(d\pi_P)_{f(q)}$

So $H_{A'}$ is a connection.

Uniqueness follows from required G-invariance.

$$(f^*\omega_{A'})(X^\#) \overset{\text{Def}^{\text{n}}}{=} \omega_{A'}(df(X^\#))$$

$X \in h$

$$df_g(X^\#) = \frac{d}{dt}\Big|_{t=0} f(ge^{tX})$$

$$= \frac{d}{dt}\Big|_{t=0} f(g)\lambda(e^{tX})$$

$$= \frac{d}{dt}\Big|_{t=0} f(g)e^{t\lambda_* X}$$

$$= (\lambda_* X)^\#_{f(g)}$$

$$= \omega_{A'}((\lambda_*(X))^\#)$$

$$= \lambda_*(X)$$

2nd formula follows from this and

$$\Omega_{A'} = d\omega_{A'} + \frac{1}{2}[\omega_{A'} \wedge \omega_{A'}]$$

$$\Rightarrow f^*\Omega_{A'} = df^*\omega_{A'} + \frac{1}{2}[f^*\omega_{A'} \wedge f^*\omega_{A'}]$$

$$= d\lambda_*\omega_A + \underline{\tfrac{1}{2}[\lambda_*\omega_A \wedge \lambda_*\omega_A]}$$

$$= \tfrac{1}{2}\lambda_*[\omega_A \wedge \omega_A]$$

bec. λ_* is a Lie alg. hom.

$$= \lambda_* \Omega_A \qquad \blacksquare$$

Notation:

A' is called λ-ext. of A

A ——————————— λ-red. of A'.

<u>Prop</u>: Let
$$\begin{tikzcd} Q \arrow[rr, "f"] \arrow[dr, "\pi_Q"'] & & P \arrow[dl, "\pi_P"] \\ & M & \end{tikzcd}$$
be a morphism of type $\lambda: H \to G$, s.th. λ_* is a Lie alg. isomorphism.

Suppose A is a connection on P.

Then $\exists$ unique connection A' (denoted by f^*A) s.th.

$$f^*\omega_A = \lambda_* \circ \omega_{A'}$$

Proof: Define

$$\omega_{A'} := \lambda_*^{-1} \circ f^*\omega_A \in \Omega^1(Q;\mathfrak{h}) \qquad \blacksquare$$

For instance this applies to

$$\mathrm{Spin}(n) \xrightarrow{2:1} SO(n)$$

$$\mathrm{Spin}^c(n) \xrightarrow{2:1} SO(n) \times S^1$$

both inducing the alg isomorphisms.

Def^n: A bundle isom on f of type id_G is called a bundle isom.

$$\begin{tikzcd} P \arrow[rr, "f"] \arrow[dr] & & P' \arrow[dl] \\ & M & \end{tikzcd}$$

In particular, $f : P \to P$ of type id is called a bundle autom.

Def^n $\mathrm{Aut}(P) = \{ f : P \to P \text{ of type id} \}$ is called the <u>gauge group</u> of P

Rk: $\mathrm{Aut}(P) \cong C^\infty_G(P;G)$

$$= \{ \varphi : P \to G \text{ s.th. } \varphi(pg) = g^{-1}\varphi(p)g = \mathrm{Ad}_{g^{-1}} \circ \varphi(p) \quad \forall p \ \forall g \ \}$$

$\longleftarrow\!\!|$

$f(p) := p\,\varphi(p)$

$f(pg) = pg\,\varphi(pg)$

$= pg\, g^{-1}\varphi(p) g = f(p)g$

$$\mathrm{Aut}(P) \cong \Gamma(M; \mathrm{Ad}(P))$$

$$\mathrm{Ad}(P) \cong P \times_{\mathrm{Ad}} G$$

not a principal G-bundle

Prop: Let $f \in \mathrm{Aut}\,P$, φ_f the ass. map $\varphi_f : P \to G$.

Then f^*A (defined by $\omega_{f^*A} := f^*\omega_A$) satisfies:

(1.) $\omega_{f^*A} = \mathrm{Ad}_{\varphi^{-1}}\,\omega_A + \varphi^{-1} d\varphi$

left mult. in the Lie group

(2.) $d_{f^*A} = f^* \circ d_A \circ (f^*)^{-1}$

(3.) $\Omega_{f^*A} = \mathrm{Ad}_{\varphi_f^{-1}} \circ \Omega_A$

Pf: Exercise [Baum: Thm 3.22]

Chern–Weil theory II

$$G \to P \xrightarrow{\pi} M \quad \text{principal}$$

$$\Phi : \underbrace{\mathfrak{g} \times \cdots \times \mathfrak{g}}_{k \text{ times}} \to \mathbb{C}$$

Ad-invariant symmetric

$$\rightsquigarrow \quad c_\Phi(A) := \Phi(\underbrace{\Omega_A \wedge \cdots \wedge \Omega_A}_{k \text{ times}})$$

$$\in \Omega^{2k}_{\substack{\text{hor.}\\ G\text{-inv.}}}(P;\mathbb{C})$$

Seen:

- $d c_\Phi(A) = 0$
- $c_\Phi(A') - c_\Phi(A) \in d\,\Omega_{\substack{\text{hor.}\\ G\text{-inv.}}}(P;\mathbb{R})$

Lemma: $\Omega^*_{\substack{\text{hor.}\\ G\text{-inv.}}}(P;\mathbb{C}) \cong \Omega(M;\mathbb{C}) \xleftarrow{\pi^*}$

$$\pi^* \overline{c_\Phi}(A) := c_\Phi(A).$$

$$[\tilde{c}_\phi(A)] =: c_\phi(P) \in H^{2k}(M;\mathbb{C})$$

$\boxed{\text{No class on } \overset{\text{Thu}}{13} \text{ May and Thu 3 June}}$

Weil-homom.

$$\boxed{\begin{array}{l} w_P : \underset{\substack{\text{alg of}\\ \text{symmetric}\\ \text{multilin.}\\ \text{forms}}}{S^*_G(\mathfrak{g})} \longrightarrow H^*_{dR}(M;\mathbb{C}) \\ \qquad\qquad \phi \longmapsto c_\phi(P). \end{array}}$$

<u>Def^n</u>: If $N \xrightarrow{f} M$

$$f^*P := \{(n,p) \in N \times P \mid f(n) = \pi(p)\}$$

$$(n,p) \longmapsto p$$

$$\begin{tikzcd} f^*P \arrow[r, "\hat{f}"] \arrow[d] & P \arrow[d, "\pi"] \\ N \arrow[r, "f"] & M \end{tikzcd}$$

is called the pull-back bundle

<u>Prop^n</u>: If A is a conn. on P, then there is a unique conn. A' on f^*P s.th.

$$\omega_{A'} = \hat{f}^*\omega_A$$

<u>Pf</u>: Define it by this formula

Check:
$$\omega_{A'}(X^\#) = \omega_A((\hat{f})_* X^\#)$$
$$= \omega_A(X^\#) = X.$$

$\blacksquare$

<u>Thm</u>[1] For the Weil hom. we have:

(1.) $c_\phi(f^*P) = f^* c_\phi(P)$ if $f: N \to M$

$\iff W_{f^*P} = f^* W_P$

(2.) If
$$\begin{tikzcd} Q \arrow[rr, "\Phi"] \arrow[dr] & & P \arrow[dl] \\ & M & \end{tikzcd}$$
ω of type $\phi \in S^*_G(\mathfrak{g})$

$\phi_\lambda := \phi \circ \lambda_*$
$\in S^*_H(\mathfrak{h})$

then

$c_{\phi_\lambda}(Q) = c_\phi(P)$

$\iff W_P^{(\phi)} = W_Q(\phi_\lambda)$.

<u>Proof</u>: (1.) follows from previous prop^n:

$$c_\phi(f^*P) = [c_\phi(\hat{f}^*A)]$$
$$= [f^* c_\phi(A)]$$
$$= f^* c_\phi(P).$$

2. by above Prop'n wrt push-forward com.

$$\begin{tikzcd} Q \arrow[rr, "f"] \arrow[dr] & & P \arrow[dl] \\ & M & \end{tikzcd}$$

f induces id on M

$\square$

<u>Rk:</u> It can be shown that any principal G-bundle admits a reduction to a maximal compact subgroup

$(U(n) \subseteq GL(n,\mathbb{C}))$

Chern classes of vector bundles

$$U(n) = \{B \in GL(n,\mathbb{C}) \mid B^*B = \mathrm{id}\}$$

$$\mathfrak{u}(n) = \{X \in \mathfrak{gl}(n,\mathbb{C}) \mid X^* = -X\}$$

skew-adjoint

<u>Def^n:</u>

$\phi_k : \mathfrak{gl}(n,\mathbb{C}) \to \mathbb{C}$

from $\det\left(t - \frac{1}{2\pi i} X\right)$ really $t \cdot \mathbb{1}_n$

$$=: \sum_{k=0}^{n} \phi_k(X)\, t^{n-k}$$

Then ϕ_k are Ad-inv't

and

$\phi_k\big|_{\mathfrak{u}(n)}$ take real values

(Pf: $\overline{\det\left(t - \frac{1}{2\pi i} X\right)} = \det\left(t\mathbb{1} + \frac{1}{2\pi i} \overline{X}^t\right)$, where $\overline{X}^t = X^*$

$$= \det\left(t - \frac{1}{2\pi i} X\right).\;)$$

Any $X \in \mathfrak{u}(n)$ is diagonalisable

$$X = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

$$\det\left(t - \frac{1}{2\pi i}\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}\right)$$

$$= \prod_{i=1}^{n} \left(t - \frac{\lambda_i}{2\pi i}\right)$$

$$\Rightarrow \Phi_1(X) = \sum \frac{-\lambda_i}{2\pi i}$$

$$= \frac{-1}{2\pi i} \operatorname{tr}(X)$$

$$\Phi_2(X) = \sum_{i<j} \lambda_i \lambda_j \left(-\frac{1}{4\pi^2}\right)$$

$$= -\frac{1}{8\pi^2} \sum_{i \neq j} \lambda_i \lambda_j$$

$$= -\frac{1}{8\pi^2}\left(\sum_{i,j} \lambda_i \lambda_j - \sum_i \lambda_i \lambda_i\right)$$

$$= \underline{\frac{1}{8\pi^2}\left(\operatorname{tr}(X^2) - \operatorname{tr}(X)\operatorname{tr}(X)\right)}$$

$\vdots$

$$\Phi_n(X) = \left(-\frac{1}{2\pi i}\right)^n \det(X)$$

<u>Thm</u> The $\Phi_0, \dots, \Phi_n \in \mathrm{Sym}^*_{U(n)}(u(n))$ are alg. indep. and generate

$$\big(\underset{\uparrow \text{polyn.}}{Q}(\Phi_0, \dots, \Phi_n) = 0 \implies Q \equiv 0\big)$$

<u>Pf</u>: $\Phi_k(X) = \left(-\frac{1}{2\pi i}\right)^k \sigma_k(\lambda_1, \dots, \lambda_n)$

k^{th} elementary sym. polyn in $\lambda_1, \dots, \lambda_n$, these have said property $\blacksquare$

Thm Let $E = P \times_{U(n)} \mathbb{C}^n$, where P is a principal $U(n)$-bundle.

Then

$$c_k(E) := [\Phi_k(F_A \wedge \dots \wedge F_A)]$$
$$\in H^{2k}_{dR}(M;\mathbb{R})$$

is the image of the Chern classes under

$$H^{2k}(X;\mathbb{Z}) \to H^{2k}(X;\mathbb{R}) \cong H^{2k}_{dR}(X)$$

$$c(E) := \left[\det\left(\mathbb{1} - \frac{1}{2\pi i} F_A\right)\right]$$
$$= \underset{\substack{\shortparallel \\ 1}}{c_0(E)} + c_1(E) + \dots + c_n(E)$$

is called the total Chern class

Pf sketch: Chern classes are characterized by the axioms:

- $c_1(E_1) = c_1(E_2)$ if $E_1 \cong E_2$
- $c(f^*E) = f^* c(E)$ if $f: N \to M$
- $c(E_1 \oplus E_2) = c(E_1) \cdot c(E_2)$

  cup. product
- $c_k(E^*) = (-1)^k c_k(E)$, $c(\underline{\mathbb{C}}^n) = 1$
- $\langle c_1(H), [\mathbb{CP}^1] \rangle = -1$

  where $H \to \mathbb{CP}^1$ is the tautological line bundle.

- We have verified $c_1(H) = -1$ last time

- $E_1 \oplus E_2$ a $GL(n_1+n_2, \mathbb{C})$-bundle has a reduction to a $GL(n_1, \mathbb{C}) \times GL(n_2, \mathbb{C})$-bundle
(resp. $U(n_1+n_2)$ to $U(n_1) \times U(n_2)$)

  Take a connection respecting this reduction and apply Thm 1.(2.)

- $c_k(E^*) = (-1)^k c_k(E)$

  E^* is associated to dual rep $U(n) \to (\mathbb{C}^n)^*$

  $(\delta_{can})_*: \mathfrak{u}(n) \to \mathfrak{gl}(n, \mathbb{C})^*$
  $X \mapsto X^* = -X$

- First two prop. also follow from Thm 1. $\boxed{}$

Example:

$$c_1(E) = -\frac{1}{2\pi i}\left[\operatorname{tr}(F_A)\right]$$

$$c_2(E) = \frac{1}{8\pi^2}\left[\operatorname{tr}(F_A \wedge F_A) - \operatorname{tr}(F_A) \wedge \operatorname{tr}(F_A)\right]$$

Rk: If we have a reduction to $SU(n)$, then

$$c_1(P) = 0$$

Pf: $\Phi_1\big|_{su(n)} = -\frac{1}{2\pi i}\operatorname{tr}\big|_{su(n)} = 0$

↑ skew-adj. traceless

$\square$

Pontryagin classes

Def^n: $E \to M$ real v.b.

$$p_k(E) := (-1)^k c_{2k}(E^{\mathbb{C}})$$

complexification $V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$, V real vector sp.

$$p(E) = 1 + p_1(E) + \dots + \in H^{4*}_{dR}(M; \mathbb{R})$$

are called Pontryagin / total Pontryagin class.

Thm These can be obtained by the above procedure from

$$\det\left(t - \frac{1}{2\pi} X\right) := \sum_{k=0}^{n} \tau_k(X) t^{n-k}$$

on $\mathfrak{gl}(n)$

and $\tau_{2k+1}\big|_{\mathfrak{o}(n)} = 0 \quad \forall k$

via $p(E) = \left[\det\left(\mathbb{1} - \frac{1}{2\pi} F_A\right)\right]$

here $\mathfrak{o}(n)$ = Lie alg of $O(n) = \{A \mid AA^t = id\}$

$\Rightarrow \mathfrak{o}(n) = \{X \mid X^t = -X\}$

Pf: Like above for Chern classes.

Notice for instance for $X \in \mathfrak{o}(n)$:

$$\mathrm{tr}(X) = \mathrm{tr}(X^t) = \mathrm{tr}(-X)$$
$$= -\mathrm{tr}(X)$$
$$\Rightarrow \mathrm{tr}(X) = 0$$

Idea: $\psi_k(X) = 0$ for k odd. $\blacksquare$