Examples:

1. On $[0,1]$ the function

$$f(x) := |\log(x)|$$

Then $f \in L^p \quad \forall p \geq 1$,

but $f \notin L^\infty$

Pf:

$$\lim_{x \to 0^+} \frac{x^\alpha}{|\log(x)|} = 0 \quad (*)$$

$\forall \alpha > 0$

$$\int_{[0,1]} |f(x)|^p \, dx$$

$$|\log(x)| \leq x^{-\alpha}$$

$\forall \alpha > 0$,

once $x \leq x_0$

for some x_0.

Choose α s.th. $-\alpha \cdot p > -1$

$$\Rightarrow \int_{[0,1]} |f(x)|^p \, dx = \underbrace{\int_{[x_0,1]} |f(x)|^p \, dx}_{=: I < \infty \;\; \forall p} + \int_{(0,x_0]} |f(x)|^p \, dx$$

$$\leq I + \underbrace{\int_{(0,x_0]} x^{-p\alpha} \, dx}_{< \infty} \qquad \blacksquare$$

2. At borderline: on B^4 we do not have $\frac{1}{2}$

$$L^4_1 \times L^4_1 \longrightarrow L^4_1$$

Counterexample:

$$f(x) := |\log(|x|)|^{1/2}$$

Pf:

- $$\int_{B^4_{1/2}} |f(x)|^4 \, dx = \int_{B^4_{1/2}} |\log(|x|)|^2 \, dx$$
$$= vol(S^3) \int_0^{1/2} |\log r|^2 \, r^3 \, dr$$
$$< \infty$$
(example 1).

- $$\frac{\partial f}{\partial r}(r) = \frac{1}{2} |\log(r)|^{-1/2} \cdot \frac{1}{r}$$

$$\Rightarrow \int_{B^4} |\nabla f|^4 \, dx$$
$$= \frac{vol(S^3)}{2} \cdot \underbrace{\int_0^{1/2} \frac{1}{(\log r)^2} \frac{1}{r} \, dr}$$
$$= \frac{vol(S^3)}{2} \left(\frac{1}{\log r} \right) \Big|_0^{1/2} < \infty \quad \boxed{}$$

But $L^4_1 \times L^4_1 \longrightarrow L^4_1$

fails

because

$$\frac{\partial}{\partial r} f^2(r) = \frac{\partial}{\partial r} \log r$$

$$= \frac{1}{r}$$

$$\Rightarrow \int_{B^4} |\nabla(f^2)|^4 dx$$

$$= \mathrm{vol}(S^3) \int_0^1 \left(\frac{1}{r}\right)^4 \cdot r^3 dr$$

$$= \mathrm{vol}(S^3) \int_0^1 \frac{1}{r} dr$$

$$= \infty$$

We will be working with

$G \subseteq$ matrix Lie group $\subseteq M(n \times n)$

$$Ad(P) \subseteq End(P \times_{G_{can}} \mathbb{R}^n)$$

$$ad(P) \subseteq End(P \times_{G_{can}} \mathbb{R}^n)$$

are subbundles

<u>Recall</u> To show $U(n)$ is a Lie group

$$\Phi : \begin{cases} M(n \times n) \longrightarrow Sym(n \times n) \\ A \longmapsto AA^* - id \end{cases}$$

show that 0 is a regular value.

$\Rightarrow U(n) = \Phi^{-1}(0)$ is a Lie group.

$\mathcal{A}^p_k := L^p_k$ - completion of $\mathcal{A}$

↑ space of connections on $P \to M$

$\mathcal{G}^p_{k+1} := L^p_{k+1}$ - completion of $\mathcal{G} = \Gamma(\mathrm{Ad}(P))$

<u>Prop^n</u>: If $w(L^p_k) = k - \frac{p}{n} > 0$, then $\mathcal{G}^p_k$ is a Banach Lie group (i.e. a Banach manifold and multipl/taking inverses are smooth maps)

<u>Pf sketch</u>:

$$L^p_k(M; \mathrm{End}(E)) \longrightarrow L^p_k(M; \mathrm{Herm}(E))$$
$$u \longmapsto uu^* - \mathrm{id}$$

is well-def. and smooth by multipl. above borderline.

Show: 0 is a regular value.

$\Rightarrow \mathrm{Ad}(P) = \Phi^{-1}(0)$ is a $L^p_k(M; \mathfrak{g})$ - Banach mfld.

Compos & inverses are smooth by Sobolev multipl. ∎

⚠ exponential map is smooth because composition

$f \in L^p_k(E)$

$\Phi : E \to F$ smooth

$\Rightarrow \Phi \circ f \in L^p_k(F)$ if $w(L^p_k) > 0$.

(compos. result)

Prop^n: $w(L^p_{k+1}) > 0$

then

$$\mathcal{G}^p_{k+1} \times \mathcal{A}^p_k \longrightarrow \mathcal{A}^p_k$$

is a smooth action $\mathcal{L}^p_{k+1} \cap C^0$

Pf: $$u(A) = A + u \underbrace{\underbrace{d_A u^{-1}}_{\in L^p_k}}_{\in L^p_k \text{ by mult.}}$$

$u \in \mathcal{G}^p_{k+1}$

$A \in \mathcal{A}^p_k$

$\blacksquare$

$\mathcal{G}^2_2$ borderline case.

$w(L^2_2) = 2 - \frac{4}{2} = 0$

$\mathcal{G}^2_2 := \{u \in L^2_2(X^4; \mathrm{End}(E)) \mid u(x) \in \mathrm{Ad}(P)_x \text{ almost everywhere}\}$

Prop $\mathcal{G}_2^2$ is a group: $u, v \in \mathcal{G}_2^2$

$$\nabla^2 u \cdot v = \underbrace{(\nabla^2 u)}_{\in L^2} \cdot v + 2 \overbrace{(\nabla u)}^{\in L_1^2 \hookrightarrow L^4} \cdot \overbrace{(\nabla v)}^{\in L_1^2 \hookrightarrow L^4} + \underbrace{u \underbrace{(\nabla^2 v)}_{\in L^2}}_{\in L^2 \text{ because}}$$

$u, v \in L^\infty$ because G is compact

By Hölder-ineq. $L^4 \times L^4 \longrightarrow L^2$

$$\Rightarrow \nabla^2 (u \cdot v) \in L^2$$

Simpler $\nabla (u \cdot v) \in L^2$

$$\Rightarrow u \cdot v \in L_2^2$$

$u \cdot v \in Ad(P)$ a.e. $\boxed{}$

<u>Prop^n</u>:

$$\mathcal{G}^2_2 \times \mathcal{A}^2_1 \longrightarrow \mathcal{A}^2_1$$

is a topological group action

<u>Pf</u>: multipl. result in borderline case. $\boxed{}$

<u>Rk</u> (Rigor)

$\mathcal{G}^2_2$ considered with the topology induced from

$$\subseteq L^2_2(X^4; \mathrm{End}(E)) \cap L^\infty.$$

Why should we care about the borderline case:

L^2_1. If A_0 is a smooth conn.

$$a \longmapsto F_{A_0+a}$$
$$= F_{A_0} + \underbrace{d_{A_0} a}_{\in L^2} + \underbrace{\tfrac{1}{2}[\overset{L^4}{\overset{\cup}{a}} \wedge \overset{L^4}{\overset{\cup}{a}}]}_{\in L^2}$$

$\Rightarrow$ $\|F_{A_0+a}\|_{L^2}$ & $\|a\|_{L^2}$ control L^2_1-norm of a

and $\|F_{A_0+a}\|^2_{L^2}$ has top'l significance for ASD connections

Prop^n: If $w(L^p_{k+1}) > 0$ $\left(\iff w(L^p_k) > -1\right)$, then

$$\mathcal{A}^p_k \longrightarrow \mathcal{A}^2_1 \quad (\text{if } k \geq 1)$$

induces an injection

$$\mathcal{A}^p_k / \mathcal{G}^p_{k+1} \longrightarrow \mathcal{A}^2_1 / \mathcal{G}^2_2 .$$

Pf idea:

Suppose $[A_1]_{\mathcal{G}^2_2} = [A_2]_{\mathcal{G}^2_2}$ for $A_1, A_2 \in \mathcal{A}^p_k$. That means $\exists\, u \in \mathcal{G}^2_2$ s.th.

$$u(A_1) = A_2 .$$

$$\iff A_1 + \underbrace{u^{-1} d_{A_1} u}_{\text{eigen (?)}} = A_2$$

$$\implies u^{-1} d_{A_1} u = \underbrace{A_2 - A_1}_{\in L^p_k}$$

$$\Rightarrow \quad d_{A_0} u = u(A_2 - A_1) - a_1 u$$

where $\left\{ \begin{array}{l} A_0 \text{ smooth} \\ A_1 = A_0 + a_1 \end{array} \right\}$

& use multipl. results

$\boxed{\text{For instance} \quad w(L^A_k) = 0}$

Multipl. of borderline

$$\Rightarrow \quad a_1 u \in L^k \quad \forall k \geq 0$$

$$\underbrace{u}_{\in L^\infty} \underbrace{(A_2 - A_1)}_{\in L^k}$$

$$\Rightarrow \quad d_{A_0} u \in L^q \quad \forall q$$

$$\Rightarrow \quad u \in L^q_1 \quad \forall q$$

$$\hookrightarrow C^0 \text{ for } q > 4.$$

Now repeat

$$d_{A_0} u = \underbrace{\overbrace{u(A_2 - A_1)}^{\in L^p_1}}_{\in L^p_1} - \overbrace{\underline{\underline{a_1 u}}}^{\in L^p_1}_{\in L^p_1}$$

by unif. above borderline ($w(L^q_1) > 0$)

$\Longrightarrow u \in L^p_2$ .....

until $u \in L^p_{k+1}$ $\blacksquare$

This technique (in the proof) is called

bootstrapping

Def[4]

$B^p_k := \mathcal{A}^p_k / \mathcal{G}^p_{k+1}$

is called the configuration space

Prop^{n}: For $w(L^{p}_{k+1}) \geq 0$, $k \geq 1$, the configuration space $\mathcal{B}^{p}_{k}$ is a Hausdorff space.

Recall:

A space X is Hausdorff iff $\Delta = \{(x,x) \mid x \in X\} \subseteq X \times X$ is <u>closed</u>.

Similarly if $G \curvearrowright X$ (finite group, space) then X/G is Hausdorff iff

$$\Gamma = \{(x, gx) \mid x \in X, g \in G\} \subseteq X \times X$$

is closed.

Proof of Prop^n:

By the previous Prop^n, it is enough to consider

$$\mathcal{B}_1^2 = \mathcal{A}_1^2 / \mathcal{G}_2^2 .$$

$$\Gamma = \{(A, u(A)) \mid A \in \mathcal{A}_1^2, u \in \mathcal{G}_2^2\}$$

Suppose $(A_i, u_i(A_i))$ is a sequence in Γ converging in L_1^2 to (A, B)

$$\left\{ \begin{array}{l} \text{Write } A_i = A_0 + a_i \\ \qquad u(A_i) = A_0 + b_i \end{array} \right. \qquad a_i, b_i \in L_1^2,$$

equivalently

$$a_i \xrightarrow{L_1^2} a := A - A_0$$

$$b_i \xrightarrow{L_1^2} b := B - A_0$$

$$w(L_1^4(X)) = 0$$

$$A_0 + b_i = u_i(A_i)$$

$$= A_0 + a_i + u_i \, d_{A_i} u_i^{-1}$$

$$= A_0 + a_i + u_i \, d_{A_0} u_i^{-1} + u_i a_i u_i^{-1}$$

$$\Rightarrow \quad u_i^{-1}(b_i - a_i) - a_i u_i^{-1} = d_{A_0} u_i^{-1}$$

$$\Leftrightarrow \quad (*) \quad d_{A_0} u_i^{-1} = \underbrace{u_i^{-1} \underbrace{(b_i - a_i)}_{\in L^2_1\text{-bounded}} - \underbrace{a_i}_{\in L^2_1\text{-bounded}} u_i^{-1}}_{L^4\text{-bounded}}$$

$$\Rightarrow \quad u_i \text{ is } L^4_1\text{-bounded}$$

by Sobolev mult. and bootstrapping:

$$(*): \quad d_{A_0} u_i^{-1} = \underbrace{\underbrace{u_i^{-1}}_{L^4_1 \cap L^\infty\text{-bounded}} \underbrace{(b_i - a_i)}_{L^2_1\text{-bdd}}}_{L^2_1\text{-bounded}} - \underbrace{a_i}_{\in L^2_1 \text{ bdd}} \underbrace{u_i^{-1}}_{L^4_1 \cap L^\infty \text{ bdd}}$$

(by $L^4_1 \hookrightarrow L^\infty$ $\times L^2_1 \longrightarrow L^2_1$
bdded)

$\Rightarrow$ $d_{A_0} u_i$ is bdded in L^2_1

$\Rightarrow$ u_i is bdded in L^2_2

Banach-Alaoglu $\Rightarrow$ $\exists$ $u \in L^2_2$ s.th.
for a subsequence $u_i \rightharpoonup u$ weakly in L^2_2

Now $L^2_2 \hookrightarrow L^2_1$ compact (Rellich-Lemma)

<u>Fact</u>: Cpt operators turn weakly conv. sequences into strongly

<u>So</u>: $u_i \to u$ in L^2_1

$\Rightarrow$ $u_i \to u$ a.e.

$\Rightarrow$ $u \in Ad(P)$ almost everywhere

It remains to show:

$$u(A) = B \in L^2_1$$

$$d_{A_0} u_i^{-1} = \underbrace{u_i^{-1}}_{\substack{\text{converges} \\ \text{in} \\ L^2_1 \cap L^\infty}} \underbrace{(b_i - a_i)}_{\substack{\to b-a \\ \text{in } L^2_1}} - a_i u_i^{-1}$$

$$\xrightarrow[\text{in } L^2]{} u^{-1}(b-a) - a u^{-1}$$

by $L^2_1 \hookrightarrow L^4$

$L^4 \times L^4 \longrightarrow L^2$ by Hölder ineq.

$$\Rightarrow d_{A_0} u^{-1} = u^{-1}(b-a) - a u^{-1}$$

$$\Rightarrow u \, d_{A_0} u^{-1} = b - a - u a u^{-1}$$

$$\Rightarrow A_0 + u \, d_{A_0} u^{-1} = A_0 + b - a - u a u^{-1},$$

$$\Rightarrow A_0 + a + u\, d_{A_0} u^{-1} = A_0 + b - u a u^{-1}$$

$$\Leftrightarrow A + u\, d_A u^{-1} = B$$

$$\Leftrightarrow u(A) = B$$

equality L^2.

$\blacksquare$