Digression: Elliptic operators

On $\mathbb{R}^n$

L a partial diff. operator of order l

$$L = P_l(D) + \dots + P_0(D)$$

$$P_j(D) = \sum_{\sum \alpha_i = j} a_\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha}$$

$$\alpha = (\alpha_1, \dots, \alpha_n)$$

For $\xi \in \mathbb{R}^n$: $\sigma(L)_j(\xi) = \sum_{|\alpha| = j} a_\alpha \xi^\alpha$ jth. „Symbol"

where

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$$

$$\xi^\alpha := \xi_1^{\alpha_1} \cdot \dots \cdot \xi_n^{\alpha_n}$$

Def^n L is called elliptic at x if $\sigma(L)(\xi)$ is invertible $\forall\, \xi \neq 0$.

L is elliptic if elliptic at any point

(The a_α's are fctns on $\mathbb{R}^n$)

Example:

1. $$\Delta = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

Then $\sigma(\Delta)(\xi) = -|\xi|^2 \cdot id$

is invertible for $\xi \neq 0$.

2. Wave operator:

On $\mathbb{R}\times\mathbb{R}^n \ni (t,x)$

$$L := \frac{\partial^2}{\partial t^2} + \Delta$$

Then $\sigma_2(L)(\xi_0, \xi^{\mathbb{R}^n})$

$$= \xi_0^2 - |\xi|^2$$

This is not elliptic.

More intrinsic def^n for

$$\begin{tikzcd} E \arrow[dr] & & F \arrow[dl] \\ & M & \end{tikzcd}$$

vector bundles on M

$L : \Gamma(E) \to \Gamma(F)$

a dif. op. of order ℓ.

Then for $\xi \in T^*_x M$, $u \in E_x$

let $\tilde{u} \in \Gamma(E)$ with $\tilde{u}(x) = u$,

$f \in C^\infty(M)$ with $f(x) = 0$

and $df_x = \xi$

$$\sigma(L)(\xi)\, u := \frac{1}{\ell!}\left(L\left(f^\ell \tilde{u}\right)\right)(x)$$

We call $\sigma(L)$ the <u>principal</u> symbol of L.

<u>Exercise</u>: This is well-defined and defines

$\sigma(L) : \mathrm{Hom}(E \otimes T^*X^{\otimes \ell}, F)$

<u>Def^n</u>: L is elliptic, if $\sigma(L)(\xi)$ is a bundle-isom. $\forall \xi \neq 0$.

<u>Exercise</u>: On $\mathbb{R}^n$ these notions coincide.

<u>Thm</u> Let $L : \Gamma(E) \to \Gamma(F)$ be elliptic of order l.
Then $\exists\, c > 0$ s.th.

$$\|u\|_{L^2_{s+l}} \leq c \cdot \left(\|Lu\|_{L^2_s} + \|u\|_{L^2_s} \right)$$

$\forall u \in L^2_{s+l}$

Remark 1. In fact we have
$\exists\, c > 0$ s.th.

$$\|u\|_{L^2_{s+\ell}} \leq c \cdot \left(\|Lu\|_{L^2_s} + \|pr_{\ker P}(u)\| \right)$$

any norm since $\ker P$ is finite dim'l see below

2. There is also a $L^p_{k+\ell}$, L^p_k - version.

Prop: If L is elliptic, then $\ker(L)$ is finite dim'l.

Pf: Suppose not $(u_n) \subseteq \ker L$, $\subseteq L^2_{s+\ell}$

We may suppose these form an L^2-orthonormal basis.

By the Thm

$$\|u_n\|_{L^2_\ell} \leq C \cdot \|u_n\|_{L^2}$$

$\Rightarrow$ (u_n) is L^2_ℓ-bounded

Rellich-Lemma $\Rightarrow$ $\exists$ subseq. converging in L^2

$\lightning$ orthonormal basis. $\blacksquare$

<u>Pf of Thm</u> Only on $T^n \cong S^1 \times \ldots \times S^1$

only for L with only ℓ^{th} order derivatives, constant coefficients.

<u>Recall</u> Plancherel:

$$L^2(T^n) \cong \ell^2(\mathbb{Z}^n)$$

$$\varphi \longmapsto (\varphi_\xi)_{\xi \in \mathbb{Z}^n}$$ Fourier coeff.

according to
$$\varphi(x) = \sum_{\xi} \varphi_{\xi}\, e^{i x \cdot \xi}$$

$$\left(\varphi_{\xi} = \frac{1}{(2\pi)^n} \int_{T^n} \varphi(x) e^{i x \cdot \xi} dx \right)$$

On $\ell^2(\mathbb{Z}^n)$ define

$$\|(\varphi_{\xi})\|_k := \sum_{\xi} |\varphi_{\xi}|^2 (1 + |\xi|^2 + \dots + |\xi|^{2k})$$

comes from an inner product

$$H_k := \{ (\varphi_{\xi}) \mid \|(\varphi_{\xi})\|_k < \infty \}$$

Then we get

$$\varphi \longrightarrow (\varphi_{\xi})$$

induces a Hilbert space isom.

$$L^2_k(T^n) \longrightarrow H_k$$

(Recall:

$\frac{\partial \varphi}{\partial x_i}$ has Fourier coeff.

$(\varphi_\xi \cdot \xi_1)_\xi$

$$\left\{ \begin{array}{l} \text{If } \varphi(x) = \sum_\xi \varphi_\xi e^{ix\cdot\xi} \\ \Rightarrow \frac{\partial \varphi}{\partial x_1} = \sum_\xi \varphi_\xi \xi_1 \cdot e^{ix\cdot\xi} \end{array} \right\}$$

If L has constant coeff.

$$\begin{tikzcd} \varphi \arrow[r, mapsto, "\text{Fourier}", "\text{coeff.}"'] \arrow[d, mapsto, "L"'] & (\varphi_\xi)_\xi \arrow[d, mapsto, "L(\xi)"] \\ L\varphi \arrow[r, mapsto, "\text{Fourier}", "\text{coeff.}"'] & (L(\xi)\varphi_\xi)_\xi \end{tikzcd}$$

multipl. with symbol

So we prove the fund. ineq. for the H_k-norms.

By ellipticity,

$$|P_\ell(\xi)\,\varphi_\xi|^2 > 0$$

$$\forall\, \xi \neq 0$$

By compactness of $S^{n-1}(T^*T^n)$

$\exists\ c > 0$ s.th.

$$|P_\ell(\xi)\,\varphi_\xi|^2 \geq c$$

$\forall\, \xi$ with $|\xi| = 1$

$\forall\, |\varphi_\xi| = 1$

$$\Longrightarrow |P_\ell(\xi)\,\varphi_\xi|^2 \geq c \cdot |\xi|^{2\ell} \cdot |\varphi_\xi|^2$$

(*) $\qquad \forall\, \xi,\ \forall\, \varphi_\xi$.

Now

$$\|L\varphi\|^2_{L^2_s} = \sum_\xi |P_\ell(\xi)\varphi_\xi|^2 \cdot (1+|\xi|^2+\dots+|\xi|^{2s})$$

$$\overset{(*)}{\geq} c \sum_\xi |\xi|^{2\ell} |\varphi_\xi|^2 (1+|\xi|^2+\dots+|\xi|^{2s})$$

$$= c \sum_\xi |\varphi_\xi|^2 (|\xi|^{2\ell}+\dots+|\xi|^{2s+2})$$

$$\Rightarrow \|L\varphi\|^2_{L^2_s} + \|\varphi\|^2_{L^2_s}$$

$$\geq 2c \sum_\xi |\varphi_\xi|^2 (1+\dots+|\xi|^{2s+2\ell})$$

$$= 2c \cdot \|(\varphi_\xi)\|_{H_{k+\ell}}$$

$$= 2c \, \|\varphi\|_{L^2_{k+\ell}} \qquad \blacksquare$$

<u>Thm</u> (Elliptic regularity)
Let $L : \Gamma(E) \to \Gamma(F)$ be elliptic of order l.
Assume $u \in L^2_s$ satisfies

$$Lu = v,$$

and $v \in L^2_t$, then

$$u \in L^2_{t+l}.$$

<u>Rk</u>: Here we should have $s \geq l$ for our def^n to apply and for $Lu \in L^2$, but there are definitions of L^2_s for any $s \in \mathbb{R}$....

<u>Cor</u>: If $Lu = v$, $v \in C^\infty$
$\Rightarrow u \in C^\infty$

Rk: Far from true for

$$L = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$$

Any $f \in \mathcal{C}^2(\mathbb{R})$ gives a solution $g_f(x,t) := f(t-x)$

to the wave equation.

Pf of elliptic regularity also use the fund. elliptic inequality.

<u>Thm</u> (Hodge decompos. thm)

Let $P: \Gamma(E) \to \Gamma(F)$ be elliptic of order ℓ, and let P^* be its formal (L^2-) adjoint. (Exercise: Then also elliptic of order ℓ, in fact $\sigma(P^*) = \sigma(P)^*$)

Then we have the following decompositions:

(i) $L^2_k(E)$

$$= \underbrace{\ker(P)}_{\subseteq L^2_k(E)} \oplus_{L^2} \operatorname{im}(P^*: L^2_{k+\ell}(F) \to L^2_k(E))$$

(ii) $L^2_{k-\ell}(F) = \ker(P^*)$

$$\oplus_{L^2} \operatorname{im}(P: L^2_k(E) \to L^2_{k-\ell}(F))$$

Pf: Let $\alpha \in L^2_k(E)$,
$\alpha \perp_{L^2} \ker(P)$.
We have to find $\omega \in L^2_{k+\ell}(F)$
s.th. $P^*\omega = \alpha$.

On $\operatorname{im}(P) \subseteq L^2_{k-\ell}$
we define

$$q(P\varphi) := \langle \alpha, \varphi \rangle_{L^2}$$

It is well defined:

If $P\varphi = P\psi \Rightarrow P(\underline{\varphi - \psi}) = 0$

$\Rightarrow \varphi - \psi \in \ker P$

$\Rightarrow \langle \alpha, \varphi \rangle_{L^2} - \langle \alpha, \psi \rangle_{L^2}$

$= \langle \alpha, \varphi - \psi \rangle_{L^2} = 0$ bec

$\alpha \perp_{L^2} \ker P$

q is bounded on $\operatorname{im}(P)$:

$$|q(P\varphi)| = |\langle \alpha, \varphi \rangle_{L^2}|$$

$$= |\langle \alpha, pr_{\ker(P)^\perp}(\varphi)\rangle_{L^2}|$$

$$\leq \|\alpha\|_{L^2} \|pr_{\ker(P)^\perp}\varphi\|_{L^2_k}$$

its proj. onto $\ker(P)$ is zero

Fund. elliptic ineq. & RE1

$$\leq c \|\alpha\|_{L^2} \cdot \|P\varphi\|_{L^2_{k-l}}$$

$\Rightarrow q|_{\operatorname{im}(P) \subseteq L^2_{k-l}}$ is bounded

Hahn-Banach theorem

$\Rightarrow q$ extends to a bounded lin. form

$$\tilde{q} : L^2(E) \longrightarrow \mathbb{R}$$

and $\|\tilde{q}\| = \|q|_{\mathrm{im} P}\|$
$\leq c \cdot \|\alpha\|_{L^2}$

Riesz' representation theorem

$\Rightarrow \exists\, u \in L^2$ s.th.

$\tilde{q}(\psi) = \langle u, \psi \rangle \quad \forall \psi \in L^2(E)$

Then we have

$$q(P\varphi) = \langle u, P\varphi \rangle$$
$$\|$$
$$\langle \alpha, \varphi \rangle \qquad \forall \varphi \in L^2_k$$

$\Rightarrow \langle u, P\varphi \rangle_{L^2} = \langle \alpha, \varphi \rangle_{L^2} \quad \forall \varphi \in L^2_k$

Formally

$\Rightarrow \langle \underbrace{P^* u}_{\in L^2_{-\ell}(E)}, \varphi \rangle_{L^2} = \langle \alpha, \varphi \rangle \quad \forall \varphi \in L^2_k$

(**)

$L^2_k \subseteq L^2$ is <u>dense</u>

so (**) implies

$$L^2_{-\ell} \ni \underline{P^* u} = \alpha$$

defined in some distributional sense

But $\alpha \in L^2_k$

By elliptic regularity,

$$P^* u = \alpha \implies u \in L^2_{k+\ell}$$

$\blacksquare$