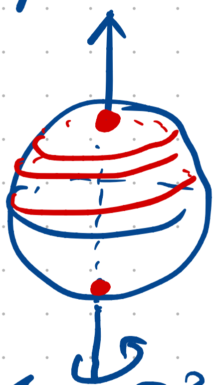
Slices to the gauge group action

In finite dimensions

$G \circlearrowright X$

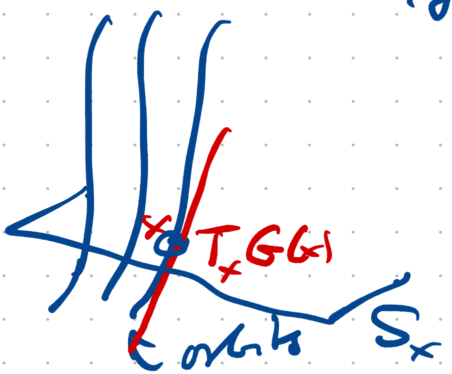
cpt Lie group

- How does X/G „look like"
- How do nbhds of orbits look like?



$S^1 \circlearrowright S^2$ by rotation

- $G(x)$ orbit of $x \in X$
  $= \{gx \mid g \in G\}$
- G_x or $\mathrm{Stab}(x)$
  $= \{g \in G \mid gx = x\}$



Slice theorem

For any $x \in X$ $\exists$ a nbhd U of 0 in $T_xX / T_x G(x)$ and a G-equiv. map

$$\Phi : G \times U / \mathrm{Stab}_x \longrightarrow X$$

which is a diffeom onto a nbhd of $G(x)$ $\quad \Phi(e,0) = x$

How to prove it:

Take an orthog. complement of $T_x G(x) \subset T_x X$ invariant under G_x (can make invariant by "averaging" over G_x).

$$\exp : T_x G(x)^{\perp} \longrightarrow X$$

$$U \longmapsto S \subseteq X$$

and then

Φ is given by

$$(g, s) \longmapsto g(s)$$

$\Uparrow$

$G \times S$.

Show that $d_{(e,0)}\Phi$ is an isom

$\Longrightarrow$ local diffeo around any $(g, 0)$ by equivariance.

Show that Φ is injective. (uses G compact).

Apply these ideas to

$\mathcal{G} \circlearrowright \mathcal{A}$

Observe:

$$T_A \mathcal{G}(A) = d_A \Omega^0(X; \mathrm{ad}(P))$$

Pf: $\xi \in \mathrm{Lie}\,\mathcal{G} = \Omega^0(X; \mathrm{ad}(P))$

$$\frac{d}{dt}\Big|_{t=0} \exp(t\xi)(A)$$

$$= \frac{d}{dt}\Big|_{t=0} \left(A + \exp(t\xi)\, d_A \exp(-t\xi)\right)$$

$$= -d_A \xi \qquad \blacksquare$$

What is the orthogonal complement ?

$$\left\{ \begin{aligned} d\,(tr(\xi \wedge * a)) &= tr(d_A \xi \wedge * a) \quad (1) \\ &\quad + tr(\xi \cdot d_A * a) \end{aligned} \right.$$

$$* \, d_A^* a$$

Integrate over X

$\int_X$ ①

$\Longrightarrow \langle d_A \xi, a \rangle_{L^2}$

$= \langle \xi, d_A^* a \rangle_{L^2} + \int_{\partial X} \mathrm{tr}(\xi \wedge {*a})$ $\stackrel{!}{=} 0$

So elements L^2-orthog. to $T_A \mathcal{G}(A)$ are given by

$\{ a \in \Omega^1(X; \mathrm{ad}(P)) \mid$

$d_A^* a = 0$ if $\partial X = \emptyset$

or

$d_A^* a = 0$ and $({*a})|_{\partial X} = 0$ if $\partial X \neq \emptyset$

$\| \ i^*({*a})$

where $i : \partial X \hookrightarrow X$

{ no constraint on ${*a}(n)$ }

$n \in TX$ normal to ∂X.

$\mathcal{G}^p_{k+1} \curvearrowright \mathcal{A}^p_k$ above the borderline. here $w(L^p_{k+1}) > 0$

$$S_{A,\varepsilon} := \{ A + a \mid d_A^* a = 0, \; {*a}|_{\partial X} = 0, \; \|a\|_{L^p_k} < \varepsilon \}$$

Hodge theorem (or analogue if $\partial X \neq \emptyset$)

$$L^p_k(X; T^*X \otimes ad(P)) = \ker(d_A : L^p_{k+1} \to L^p_k) \oplus \{ a \in L^p_k \mid d_A^* a = 0, \; {*a}|_{\partial X} = 0 \}$$

(have to solve a boundary value problem if $\partial X \neq \emptyset$)

We have proved this last time for $p=2$:

$$\Omega^k(X) = \operatorname{Im}(d) \oplus \operatorname{Im}(d^*)$$

$$\begin{cases} \text{in appr. } L^2_a\text{-completions} \\ \Delta := (d+d^*)^2 \text{ is elliptic} \end{cases}$$

We consider

$$m: \begin{cases} \mathcal{G}^p_{k+1} \times S_{A,\varepsilon} \longrightarrow \mathcal{A}^p_k \\ (g, A+a) \longmapsto g^*(A+a) \end{cases}$$

$$= g \cdot d_{A+a} \circ g^{-1}$$

Now

$$d_{(e,A)} m: \begin{cases} L^p_{k+1}(\mathrm{ad}\,P) \oplus \{a \in L^p_k \mid d^*_A a = 0,\ *a|_{\partial X} = 0\} \\ (\xi, a) \longmapsto -d_A \xi + a \end{cases}$$

and this is surjective by the Hodge decomposⁿ thm.

<u>Prop</u>:

$$\bar{m}: \Omega^p_{k+1} \times S_{A,\varepsilon} / \mathrm{Stab}(A) \longrightarrow \mathcal{A}^p_k$$

is a local difeom onto its image if ε is small enough

<u>Pf</u>: $(d_{(e,A)} m)(\xi, a) = 0$

$\Longleftrightarrow -d_A \xi + a = 0$

$\overset{\text{Hodge}}{\underset{\text{decompⁿ}}{\Longleftrightarrow}} d_A \xi = 0$ and $a = 0$

$\Longleftrightarrow \xi \in T_e \mathrm{Stab}(A)$

$\Rightarrow d_{(e,A)} \bar{m}$ is an isom

By equiv.

$d_{(g,A)}\bar{m}$ is an isom.

$\forall g \in \mathcal{G}$.

Implicit fcn thm $\Longrightarrow$ $\exists$ Stab(A)-invariant nbhd V of $\mathrm{Stab}(A) \subseteq \mathcal{G}$

and some $\varepsilon > 0$ s.t.

$$\bar{m} : V \times S_{\varepsilon}/\mathrm{Stab}(A) \longrightarrow \mathcal{A}^p_k$$

is a local difeom. near $[(e, 0)]$. Equiv.

$\Rightarrow$ same $\varepsilon > 0$ and $g(V)$ works for all $[(g, 0)]$

$\blacksquare$

For a global diffeo onto image, need to show that $\bar{\pi}$ is injective.

$$0 = d_A(g^{-1} g)$$
$$= (d_A g^{-1}) g + g^{-1} d_A g$$
$$\Rightarrow d_A g^{-1} = -g^{-1} d_A g \, g^{-1}$$

(*1)

Prop^n: Let $n = \dim X \geq 2$.

Then $\exists\, \varepsilon > 0$ s.th.

$\overline{m}$ is injective.

Proof: Assume

$$g(A+a) = g'(A+b)$$

can absorb

wlog

$$g(A+a) = A+b.$$

for $A+a$, $A+b \in S_{A,\varepsilon}$.

$$\iff g \circ d_{A+a} \circ g^{-1} = d_{A+b}$$

$$\iff g(d_A g^{-1}) + g a g^{-1} = b.$$

$$\iff \boxed{d_A g = ga - bg} \quad (*)$$

$$d_A^{*} = - * d_A *$$

We apply d_A^*

$$\Rightarrow \quad (*) \left[\begin{aligned} d_A^* d_A g &= *\big((d_A g) \wedge *a\big) \\ &\quad + *\big(*b \wedge d_A g\big) \\ &=: \Psi(d_A g, a) \\ &\quad - \Phi(b, d_A g) \end{aligned}\right.$$

Ψ, Φ bilinear.

{ Lemma: $\exists\, C > 0$ s.t.

$$\| d_A^* d_A g \|_{(L_1^2)^*} \geq C \cdot \| d_A g \|$$

for all $g \in L_1^2(X; \mathrm{End}(E))$.

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where $d_A g = ga - bg$

Assume Lemma for now.

(informally $d_A^* |_{\mathrm{im}\, d_A}$ is injective)

Recall:

$$(L^p)^* = L^q$$

where

$$\frac{1}{p} + \frac{1}{q} = 1$$

via integration:

$$L^q \to (L^p)^*$$

$$f \mapsto (h \mapsto \int hf)$$

Also

$$L^2_1 \hookrightarrow L^{\frac{2n}{n-2}}$$

$\uparrow$ w $= 1 - \frac{n}{2}$ and $\uparrow$ w $= -\frac{n}{\frac{2n}{n-2}} = 1 - \frac{n}{2}$

$\Rightarrow$

$$(L^{\frac{2n}{n-2}})^* \hookrightarrow (L^2_1)^*$$

$$L^2 \times L^n \hookrightarrow (L^{\frac{2n}{n-2}})^*$$

is bounded bilinear (induced from integration)

because

$$\frac{1}{2} + \frac{1}{n} + \frac{n-2}{2n} = 1$$

$\Rightarrow$ We get

$$L^2 \times L^n \hookrightarrow (L^2_1)^*$$

From (*) and Lemma, we get

$$\|d_A g\|_{L^2} \leq \frac{1}{C} \|d_A^* d_A g\|_{(L^2_1)^*}$$

$$\overset{(*)}{\leq} \frac{1}{C}\Big(\|\Phi(d_A g, a)\|_{(L^2_1)^*}$$

$$+ \|\Phi(b, d_A g)\|_{(L^2_1)^*}\Big)$$

$$\leq C_1 \Big(\|d_A g\|_{L^2}(\|a\|_{L^4} + \|b\|_{L^4})\Big)$$

Therefore we get:

$$\|d_A g\|_{L^2}(1 - C_1 \|a\|_{L^4} + \|b\|_{L^4}) \leq 0$$

We can conclude

$d_A g = 0$ if $\underbrace{(1 - C_1 \|a\|_{L^4} \|b\|_{L^4})}_{>0}$

but $\quad L^p_k \hookrightarrow L^n$

$\rotatebox{90}{$\sim$}$ $\qquad\qquad$ $\rotatebox{90}{$\sim$}$

$= k - \frac{n}{p} > -1 \qquad\qquad = -\frac{n}{n} = -1$

Therefore for some $\varepsilon > 0$ we get $\left(1 - C_1 \|a\|_{L^n} \|b\|_{L^n}\right) > 0$

if $\|a\|_{L^p_k}, \|b\|_{L^p_k} < \varepsilon$.

∎

We have proved the following

Theorem („Big slices")

$$S^{big}_{A,\varepsilon} = \{ A + a \mid a \in L^p_k,\ d_A^* a = 0,\ *a|_{\partial X} = 0,\ \|a\|_{L^u} < \varepsilon \}$$

Then for suff. small $\varepsilon > 0$,

$$\bar{\pi} : \mathcal{G}^p_{k+1} \times S^{big}_{A,\varepsilon} / \mathrm{Stab}(A) \longrightarrow \mathcal{A}^p_k$$

is a diffeom onto its image

Addendum: The previous map $\bar{\pi}$ is open in the (weaker) L^u-topology on $\mathcal{A}^p_k$

(see: [Taubes, connectedness of the space of connections...])

Rk: Pf of Thm used $n>2$, but also works for $n=2$: [Mrowka–Wehrheim]

Pf of Lemma:

$g \in L_1^2$, $a, b \in L_1^{n/2}$

{ Informally:

$$\Omega^0 \underset{d_A^*}{\overset{d_A}{\rightleftarrows}} \Omega^1$$

d_A^* is injective on $\mathrm{im}(d_A)$. }

$$d_A g = ag - gb \quad \text{holds in } L^2$$

get

$$L_1^{n/2} \times L_1^2 \to L^{\frac{2n}{n+2}}$$

(below borderline)

$$\Rightarrow d_A g \in L_1^{2n/n+2}$$

{ $w(L_1^{n/2}) = -1 \geq w(L^2) = -\frac{n}{2}$

$w(L_1^2) = 1 - \frac{n}{2} \leq 0$ }

Rk: $L^2(X) \xrightarrow{\text{restr.}} L^2(\partial X)$

<u>not</u> well defined

but there is a bounded map

$$L^p_k(X) \longrightarrow L^p_{k-\frac{c}{p}}(M)$$

" trace theorem "

where $M \hookrightarrow X$ has codimension c

Rk $\Longrightarrow$ $d_A g|_{\partial X} \in L^p_{1-\frac{1}{p}}$ is well-defined

$p = ?$ where

$$\Longrightarrow \boxed{* d_A g|_{\partial X} = 0} \quad (*)$$

$$*a|_{\partial X} = 0, \quad *b|_{\partial X} = 0$$