__Theorem__ („Big slices")

$$S^{big}_{A,\varepsilon} = \{ A+a \mid a \in L^p_k,\ d_A^* a = 0,\ *a|_{\partial X} = 0,\ \|a\|_{L^n} < \varepsilon \}$$

Then for suff. small $\varepsilon > 0$,

$$\bar{m} : \mathcal{G}^p_{k+1} \times S^{big}_{A,\varepsilon} / \mathrm{Stab}(A) \longrightarrow \mathcal{A}^p_k$$

is a diffeom onto its image

__Addendum__: The previous map $\bar{m}$ is open in the (weaker) L^n-topology on $\mathcal{A}^p_k$

(see: [Taubes, connectedness of the space of connections...]

<u>Rk</u>: Pf of Thm used $n>2$, but also works for $n=2$. [Bronzke-Wehrheim]

<u>Pf of Lemma:</u>

$g \in L^2_1$, $a, b \in L^{n/2}_1$

{Informally: $\Omega^0 \underset{d_A^*}{\overset{d_A}{\rightleftarrows}} \Omega^1$ d_A^* is injective on $\operatorname{im}(d_A)$.

$d_A g = ag - gb$ holds in L^2

get $L^{n/2}_1 \times L^2_1 \to L^{\frac{2n}{n+2}}$ (below borderline)

$\Rightarrow d_A g \in L_1^{2n/n+2}$

$\left\{ \begin{aligned} w(L^{n/2}_1) &= -1 \\ &\geq w(L^2) \\ &= -\frac{n}{2} \\ w(L^2_1) &= 1-\frac{n}{2} \\ &\leq 0 \end{aligned} \right.$

Rk:

$$L^2(X) \xrightarrow{\text{restr.}} L^2(\partial X)$$

not well defined

but there is a bounded map

$$L^p_k(X) \longrightarrow L^p_{k-\frac{c}{p}}(M)$$

$$\boxed{\text{„trace theorem"}}$$

where $M \hookrightarrow X$ has codimension c

Rk $\Longrightarrow$ $d_A g|_{\partial X} \in L^p_{1-\frac{1}{p}}$ is well-defined

$p = ?$

where

$$\Longrightarrow \boxed{*d_A g|_{\partial X} = 0} \quad (*)$$

$$*a|_{\partial X} = 0, \quad *b|_{\partial X} = 0$$

$$\| d_A^* d_A g \|_{(L^2_1)^*}$$

$$= \sup_{\|v\|_{L^2_1} = 1} \langle d_A^* d_A g , v \rangle_{L^2}$$

$$\overset{\circledast}{=} \sup_{\|v\|_{L^2_1} = 1} \langle d_A g , d_A v \rangle_{L^2}$$

boundary terms vanish.

(**)

Decompose

$g = g_1 + g_2$ (as sections of $\mathrm{End}(E)$) where

$g_1 \in \ker d_A$

$g_2 \in (\ker d_A)^{\perp_{L^2}}$

(***)

$$\|d_A g\|_{L^2} = \|d_A g_2\|_{L^2}$$

$$\geq C \cdot \|g_2\|_{L^2_1} \quad \text{for some } C > 0$$

(see fund. elliptic inequality $\sigma_\xi(d_A)$ is injective)

Apply (**) with

$$v = \frac{g_2}{\|g_2\|_{L^2_1}}$$

Then

$$\|d_A^* d_A g\|_{(L^2_1)^*}$$

$$\geq \frac{1}{\|g_2\|_{L^2_1}} \cdot \langle d_A g, d_A g \rangle_{L^2}$$

$$\overset{(\ast\ast\ast)}{\geq} \frac{C}{\|d_A g\|_{L^2}} \|d_A g\|_{L^2}^2$$

$$= C \cdot \|d_A g\|_{L^2} \qquad \blacksquare$$

Thm (Uhlenbeck's fundamental Lemma)

Let B^n be the flat n-dim'l unit disc. ($n \leq 4$).

$\exists\, C, \varepsilon > 0$ s.th. the foll. holds:

For all L^2_1 - connections on (the trivial bundle) on B

$$\int_{B^n} |F_A|^2 \mathrm{vol} \leq \varepsilon$$

$\exists$ gauge transf. $g \in \mathcal{G}^2_2$ s.th.

$$g^*(A) = \Gamma + a$$

↑ trivial conn.

and

→

(1.) $d_\Gamma^* a = 0$

(2.) $*a|_{\partial B^4} = 0$

(3.) $\int_{B^4} (|\nabla_\Gamma a|^2 + |a|^2)\,vol$

$$\leq C \cdot \int_{B^4} |F_A|^2 \, vol$$

„L^2_1-norm of a is controlled by $\|F_A\|_{L^2(B^4)}$"

inv. under gauge tr.

Rk: „Bubbling" shows that „this" won't work $\forall \varepsilon > 0$ (curv. bd. is essential)

Key lemma:

$\exists\, C_1, \varepsilon_1 > 0$ s.th.
$\forall\, A = \Gamma + b$ with

- $\|b\|_{L^4} \leq \varepsilon_1$
- $d^* b = 0$
- $*b|_{\partial B} = 0$

one has
$$\|b\|^2_{L^2_1(B)} \leq C_1 \cdot \|F_A\|^2_{L^2(B)}$$

Pf of key lemma:

Use Weitzenböck formula on 1-forms:

$$\Delta = \nabla^*_\Gamma \nabla_\Gamma + K$$

$\Delta = (d+d^*)^2 = dd^* + d^*d$

$\nabla^*_\Gamma \nabla_\Gamma$: connection laplacian

K: curvature term: Ricci curvature of Riem. mfld.

(see literature)

$$\int_B |F_A|^2 \mathrm{vol} = \int_B |db + b\wedge b|^2 \mathrm{vol}$$

triangle inequality

$$\geq \int_B \left(|db|^2 - |b|^4\right) \mathrm{vol}$$

$$= \int_B |\nabla_\Gamma b|^2 \mathrm{vol} - \|b\|_{L^4}^4$$

integrate the Weitzenböck-formula, $K \equiv 0$ on B and 2-forms vanish $*b|_{\partial X} = 0$

$$\geq \int_B |\nabla_\Gamma b|^2 \mathrm{vol} + \varepsilon_1^2 C_{Sob}^2 \|b\|_{L_1^2}^2$$

here

$$\|b\|_{L^4} \leq C_{Sob} \|b\|_{L_1^2}$$

Claim: $\exists\, \lambda_1 > 0$ s.th.

$$\int_B |\nabla_\Gamma b|^2 \mathrm{vol} \geq \lambda_1^2 \int_B |b|^2 \mathrm{vol}$$

So assuming claim

$$\Rightarrow \int_B |F_A|^2 \mathrm{vol}$$

$$\geq \underbrace{\left(\frac{\lambda_1^2}{1+\lambda_1^2} - \varepsilon_1^2 C_{sob}^2 \right)}_{=: C(\varepsilon_1)} \|b\|_{L_1^2}$$

because

$$\|b\|_{L_1^2}^2 = \|b\|_{L^2}^2 + \|\nabla_\Gamma b\|_{L^2}^2$$

$$\overset{\text{claim}}{\leq} \underbrace{\left(\frac{1}{\lambda_1^2} + 1 \right)}_{= \frac{\lambda_1^2 + 1}{\lambda_1^2}} \|\nabla_\Gamma b\|_{L^2}^2$$

Once we make ε_1 small enough,

$C(\varepsilon_1)$ becomes

positive. $\blacksquare$

Pf of claim:

$$\inf_{\substack{\|b\|_{L^2}=1 \\ *\partial b|_{\partial B}=0}} \|\nabla_\Gamma b\|_{L^2} =: \lambda_1 > 0$$

Suppose $\lambda_1 = 0$

$\Rightarrow \exists (b_n) \subseteq L^2_1$ with $\|\nabla_\Gamma b_n\|_{L^2} \to 0$

$\Rightarrow \exists$ subseq. $b_n \to b$ (weakly) in L^2_1

$\Rightarrow b_n \to b$ in $L^2_{1/2}$

$\overset{\text{trace}}{\underset{\text{thm}}{\Longrightarrow}}$ $*b_n|_{\partial B} \longrightarrow *b|_{\partial B}$ in $L^2(\partial B)$

$*b_n|_{\partial B} \equiv 0$ by ass.

$$\nabla_\Gamma b \equiv 0 \qquad (?)$$

Boundary value problem

$$\Longrightarrow b \equiv 0 .$$

$$\nabla_\Gamma b_n \rightharpoonup \nabla_\Gamma b \qquad \text{weakly in } L^2$$

but we have

$$\nabla_\Gamma b_n \longrightarrow 0 \qquad \text{in } L^2$$

So $\nabla_\Gamma b_n \rightharpoonup 0$

Uniqueness of weak limit $\Longrightarrow$ $\blacksquare$

In a Hilbert space H

(e_n) ONB. Then $e_n \rightharpoonup e$

i.e. $\ell(e_n) \to \ell(e)$

$\forall \ell \in H^*$

Pf of Uhlenbeck's thm:

by „method of continuity"

$$V_\varepsilon = \{ A \ L^2_1\text{-conn on } B^4 \mid \|F_A\|^2_{L^2} \le \varepsilon \}$$

$V_\varepsilon \supseteq W^C_\varepsilon := \{ A \mid$ conclusion of the Thm holds with constant $C \}$

<u>We show:</u> $\exists\, \varepsilon, C > 0$ s.th.

(1.) V_ε is connected

(2.) W^C_ε is closed

(3.) W^C_ε is open

(4.) $W_\varepsilon \neq \emptyset$.

Then we get $W_\varepsilon = V_\varepsilon$

(1.) $A = \Gamma + a \in V_\varepsilon$.

$A_t := \Gamma + r_t^* a$

where $r_t : \begin{cases} B \to B \\ x \mapsto tx \end{cases}$

$$\int_B |F_{A_t}|^2 \mathrm{vol} = \int_B |r_t^* F_A|^2 \mathrm{vol}$$

$$\overset{\text{transformation}}{\underset{\text{rule}}{=}} \int_{r_t(B)} |F_A|^2 \mathrm{vol}$$

$$\leq \int_B |F_A|^2 \mathrm{vol} \leq \varepsilon$$

A_t is a path connecting A to Γ

(4.) $\Gamma \in W_\varepsilon$ trivially.

(2.) Let (A_i) be a sequence in W_ε^C, converging in L^2_1 to some connection A.

$\exists\, u_i \in \mathcal{G}_2^2$ s.th.

$$u_i^*(A_i) =: \Gamma + a_i$$

with

$$\begin{cases} d_\Gamma^* a_i = 0 \\ *a_i|_{\partial B} = 0 \\ \|a_i\|^2_{L^2_1} \leq C \cdot \|F_{A_i}\|^2_{L^2} \end{cases}$$

As in proof that $\mathcal{B}_1^2 = \mathcal{A}_1^2/\mathcal{G}_2^2$ is Hausdorff, we get

$$u_i \longrightarrow u \in \mathcal{G}_2^2 \quad \text{in } L^2_2$$

$$a_i \longrightarrow a \quad \text{in } L^2_1$$

s.th. $u(A) = \Gamma + a$.

The estimates hold in the limit because

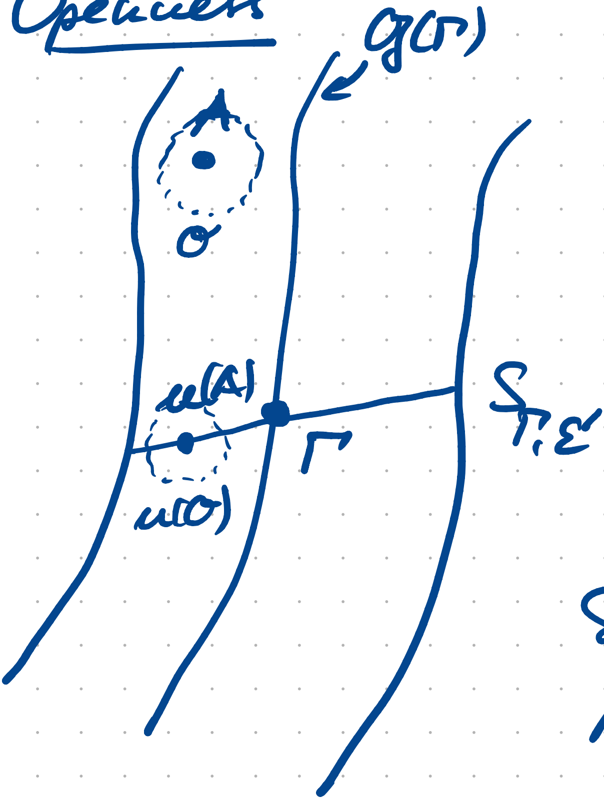
convergence is in L^2_1

and $L^2_1(B^4) \xrightarrow{\text{rstr.}} L^2_{1/2}(\partial B)$

$\hookrightarrow L^2(\partial B^4)$

is bounded.

(3) <u>Openness</u>



Big slice thm

Suppose $A \in W_\varepsilon^{\,c}$

δ: size of L^2_1-nbhd of $g(\sigma)$ in big slice thm for L^2_1

Show: $\exists\, \eta > 0$ s.th. for any $a' \in L^2_1$ with $\|a'\|_{L^2_1} < \eta$

$\exists\, u' \in \mathcal{G}^2_2$ s.th.

$u'(A + a')$ satisfies the conclusion.

Suppose u works for A.

$$\Rightarrow u(A + a') = u(A) + u a' u^{-1}$$

$$\|u a' u^{-1}\|_{L^4} \le \|a'\|_{L^4} \le C_{Sob} \|a'\|_{L^2_1} < C_{Sob} \cdot \eta$$

$$\Rightarrow \|\nabla_\Gamma (u a' u^{-1})\|_{L^2} \le \|u\|_\infty \|a'\|_{L^2_1} + 2\|a'\|_{L^4} \|du\|_{L^4}$$

can be made small by taking $\|a'\|_{L^2_1}$ small.

$$u(A) = \Gamma + b$$

$$\Rightarrow \|b + u a' u^{-1}\|_{L^2_1}$$

$$\le \underbrace{\|b\|_{L^2_1}}_{\le C\cdot\varepsilon} + \underbrace{\|u a' u^{-1}\|_{L^2_1}}_{\text{as small as we want by taking } \eta \text{ small}}$$

$$\overset{!}{<} \delta$$

Slice then applies w.r.t.

$\Rightarrow u_1 \in \mathcal{G}_2^L$ s.th.

$$u_1(u(A+a')) =: A + b'$$

$$\in S_{\Gamma,\delta}^{L^2_1\text{-big}}$$

$\Rightarrow$ b' satisfies (1.)(2.)

Missing step: By choosing ε small enough we can assume $\|b'\|_{L^4} \le \varepsilon_1$, which appears in key lemma.

(Apparently

$$2 C_1 C_{Sob} \varepsilon \leq \varepsilon_1$$

then the conclusion holds)

Details: Donghao Wang's notes from Tom Mrowka's class.