The Hodge star

(M, g) oriented Riem. mfld

$$g : TM \times TM \to \mathbb{R}$$

$$\rightsquigarrow g^* : T^*M \times T^*M \to \mathbb{R}$$

which is such that

$$TM \to T^*M$$
$$v \mapsto g(v, -)$$

becomes an isom.

g^* extends to tensor powers (sym / antisym)

Def^n + Prop^n: $n = \dim M$

There is a unique linear map

$$\Lambda^k T^*M \longrightarrow \Lambda^{n-k} T^*M$$
$$\omega \longmapsto *\omega$$

s.th. $\omega \wedge \mu = \langle *\omega, \mu \rangle_{g^*} \cdot \mathrm{vol}_g$

holds for all $\omega \in \Lambda^k T^*M$, $\mu \in \Omega^{n-k}$

Pf: 1-dim. gen. by vol_g

$$\Lambda^k \times \Lambda^{n-k} \longrightarrow \Lambda^n$$
$$(\omega, \mu) \longmapsto \omega \wedge \mu$$

is a non-deg. pairing

Dualizing by vol

$$\ell : \omega \longmapsto \underbrace{\left(\mu \longmapsto \frac{\omega \wedge \mu}{vol_g}\right)}_{\in (\Lambda^k T^*M)^*}$$

Metric defines an isom

$$(\Lambda^k T^*M)^* \cong \Lambda^k T^*M$$
$$\langle \nu, - \rangle_{g^*} \longleftarrow\!\shortmid \ \nu$$

$*\omega$ def. as the unique ν s.th.

$$\ell(\omega) = \langle \nu, - \rangle_{g^*}.$$

$\nu := *\omega$ $\square$

Prop^n: $*$ defines an isometry.

Prop^n: Let $(e_1, \dots, e_n)$ be an oriented orthonorm. basis of TM, $(e^1, \dots, e^n)$ the dual basis. Then

$$\mathrm{vol}_g = e^1 \wedge \dots \wedge e^n$$

and

$$\boxed{* e^I = e^J},$$

where $I \subseteq \{1, \dots, n\}$

$$J = \{1, \dots, n\} \setminus I$$

$$e^I := e^{i_1} \wedge \dots \wedge e^{i_k}$$

if $I = \{i_1, \dots, i_k\}$

$$i_1 < \dots < i_k$$

Pf: $e^I \wedge e^J = \mathrm{vol}_g$

$$= e^1 \wedge \dots \wedge e^n. \quad \square$$

Notice:

$$\omega \wedge * \omega = |\omega|^2_{g^*} \cdot \mathrm{vol}_g$$

Prop: $* \circ * = (-1)^{k(n-k)} \cdot \mathrm{id}$

Example: $n=4$
$(e^1, \dots, e^4)$ ONB

$$*e^2 = -e^1 \wedge e^3 \wedge e^4$$
$$*e^1 \wedge e^2 = e^3 \wedge e^4$$
$$*e^1 \wedge e^3 = -e^2 \wedge e^4$$
$$*e^1 \wedge e^4 = e^2 \wedge e^3$$

Obs $*^2 = \mathrm{id}$ on $\Lambda^2 T^*M$

$$\Rightarrow \Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$$

eigenspace decompos.

Both are 3-dim'l.

$$\underbrace{\begin{matrix} e^1 \wedge e^2 + e^3 \wedge e^4 \\ e^1 \wedge e^4 + e^2 \wedge e^3 \\ e^1 \wedge e^3 - e^2 \wedge e^4 \end{matrix}}_{\text{forms a basis of } \Lambda^2_+} \qquad \underbrace{\begin{matrix} e^1 \wedge e^2 - e^3 \wedge e^4 \\ e^1 \wedge e^4 - e^2 \wedge e^3 \\ e^1 \wedge e^3 + e^2 \wedge e^4 \end{matrix}}_{\text{of } \Lambda^2_-}.$$

Exercice:

Under a conformal change of the metric

$$g \longmapsto t^2 \cdot g \qquad t \in C^\infty(M)$$

Then

$$*_g\big|_{\Lambda^k} \longmapsto t^{n-2k} \cdot *_{t^2 g}\big|_{\Lambda^k}$$

In particular, for $n = 2k$, $*\big|_{\Lambda^k}$ is conformally invariant

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We had for A a conn. in P, $E = P \times_G V$ ~~...~~

$$d_A : \Omega^k(M;E) \to \Omega^{k+1}(M;E)$$

Def^n:

$$d_A^* := (-1)^{nk+1} * \circ\, d_A \circ *$$

on $\Omega^k(M;E)$.

is called the co-differential or formal adjoint of d_A.

$$d_A^*: \Omega^k(M;E) \xrightarrow{*} \Omega^{n-k}(M;E) \xrightarrow{d_A} \Omega^{n-k+1}(M;E) \xrightarrow{*} \Omega^{k-1}(M;E).$$

(up to sign)

Suppose
$(\cdot,\cdot): V \times V \to \mathbb{R}$
is an G-inv. inner product

$\rightsquigarrow$ get $(\cdot,\cdot) \otimes g^*$ on $E \otimes \Lambda^k T^*M$
inner product

For $\mu, \omega \in \Omega^k(M;E)$ $\swarrow$ $(\cdot,\cdot)$ suppress

$$\langle \omega, \mu \rangle_{L^2} := \int_M *\omega \wedge \mu$$

$$\left(= \int_M (*\omega \overset{\wedge}{,} \mu) \right)$$

explicit notation

Prop: If M is closed, then we have

$$\langle d_A \omega, \eta \rangle_{L^2} = \langle \omega, d_A^* \eta \rangle$$

for all $\omega \in \Omega^{k-1}(M;E)$
$\eta \in \Omega^k(M;E)$

(justifying "formal adjoint").

Pf:

$$\langle d_A \omega, \eta \rangle_{L^2} = \int_M (*d_A \omega) \wedge \eta$$

$*$ is an isom.

$$= \int_M (d_A \omega) \wedge *\eta$$

Leibniz rule

$$= \int_M d(\omega \wedge *\eta) - (-1)^{k-1} \int_M \omega \wedge d_A *\eta$$

ordinary d

$$\overset{\text{Stokes' Thm}}{=} -(-1)^{k-1} \int \omega \wedge d_A *\eta$$

$$= (-1)^{k-1}(-1)^{k-1}(-1)^{i-(k-1)} \int_M \omega \wedge *(*d_A * \eta)$$

$$= \int_M \omega \wedge * d_A^* \eta$$

$$= \langle \omega, d_A^* \eta \rangle_{L^2} \quad \blacksquare$$

Also works for non-closed mflds if dif. forms have compact support

Rk:

$*$ extends to

$$\Lambda^k T^*M \otimes E \to \Lambda^{n-k} T^*M \otimes E$$

(Anti) selfdual connections

Now M = oriented Riem. 4-mfld

$$\begin{array}{c} P \\ \downarrow \\ M \end{array} \qquad F_A \in \Omega^2(M; ad(P))$$

has a decompos^n

$$F_A = F_A^+ + F_A^-$$

acc. to $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$

Def^n A G-connection on $P \to M^4$ is called an (anti-)selfdual instanton if

$$F_A^+ = 0 \quad \text{(anti-selfdual)}$$

(resp. $F_A^- = 0$ (selfdual instanton))

Yang-Mills function(al)

$$\mathcal{E}(A) := \|F_A\|^2_{L^2(M)}$$

$$= -\int_M tr(F_A \wedge * F_A)$$

if A is a $u(n)$-connection

(later only $u(2)$, $su(2)$ or $SO(3) = PU(2)$-connection...)

$(\cdot,\cdot)$ $gl(n,\mathbb{C}) \times gl(n,\mathbb{C}) \to \mathbb{C}$

$(X,Y) \longmapsto tr(\bar{X}^t Y)$

is a cpx inner product

$gl(n,\mathbb{C}) = u(n) \otimes_{\mathbb{R}} \mathbb{C}$

On $u(n)$ we have $\bar{X}^t = -X$

$(\cdot,\cdot)|_{u(n)}$ is real and ad-invariant.

<u>Propⁿ</u>:

$$\mathcal{E}(f^*A) = \mathcal{E}(A)$$

for $f \in \mathrm{Aut}(P)$ gauge-transform.

<u>Pf</u>: $f^*\Omega_A = \Omega_{f^*A}$

$$= \mathrm{ad}_{\sigma_f} \circ \Omega_A$$

where $\sigma_f : P \to G$ Ad-equiv. ass. to f

<u>Propⁿ</u>: A critical pt of $\mathcal{E}$ is given by a solⁿ of

$$d_A^* F_A = 0$$

(called the Yang-Mills eqn)

Pf:

$$F_{A+ta} = F_A + t\,d_A a + \frac{t^2}{2}[a \wedge a]$$

for $a \in \Omega^1(M; \mathrm{ad}(P))$

$$\frac{d}{dt}\Big|_{t=0} \mathcal{E}(A+ta) \qquad (*)$$

$$= \langle F_A, d_A a\rangle_{L^2} + \langle d_A a, F_A\rangle_{L^2}$$

$$= 2\langle a, d_A^* F_A\rangle_{L^2}.$$

So $d_A^* F_A = 0 \iff (*) = 0 \;\forall a$. $\square$

Prop^n: If A is an (anti-) selfdual connection on $P \to M^4$, then it solves the Yang-Mills eqn.

Pf: $d_A^* = \pm * d_A *$

$$\Rightarrow d_A^* F_A = \pm * d_A \underbrace{* F_A}_{= \pm F_A}$$

because A is (anti-)selfdual

$$= (\pm 1) * \underbrace{d_A F_A}_{=0}$$

by Bianchi identity. $\blacksquare$

Suppose now A is a $U(n)$-connection (or $su(n)$-conn.) on $E \to M$

$$\mathcal{E}(A) = -\int_M tr(F_A \wedge * F_A)$$

$$= -\int_M tr(F_A^- \wedge * F_A^-) - \int_M tr(F_A^+ \wedge * F_A^+)$$

Exercise! $\Lambda^2_+ \oplus \Lambda^2_-$ is g-orthogonal

$$= \|F_A^-\|_{L^2}^2 + \|F_A^+\|_{L^2}^2$$

Globe: Chern-Weil theory

$$8\pi^2 c_2(P) = \int_M \mathrm{tr}(F_A \wedge F_A)$$

same exercise

$$= \int_M \mathrm{tr}(F_A^+ \wedge F_A^+) + \int_M \mathrm{tr}(F_A^- \wedge F_A^-)$$

$$= \int_M \mathrm{tr}(F_A^+ \wedge *F_A^+) - \int_M \mathrm{tr}(F_A^- \wedge *F_A^-)$$

$$= -\|F_A^+\|_{L^2}^2 + \|F_A^-\|_{L^2}^2$$

Comparison

$$\Longrightarrow \mathcal{E}[A] = \begin{cases} 8\pi^2 c_2(P) + 2\|F_A^+\|_{L^2}^2 \\ -8\pi^2 c_2(P) + 2\|F_A^-\|_{L^2}^2 \end{cases}$$

Prop^n: $\mathcal{E}[A] \geq 8\pi^2 \cdot |\langle c_2(P), [M]\rangle|$

and we have

(1.) If $c_2(P) > 0$ equality holds iff A is ASD and there are no self-dual conn.

(2.) If $c_2(P) < 0$ equality holds iff A is SD and there are no ASD conn.

(3.) If $c_2(P)=0$ equality holds iff A is flat. $\blacksquare$

Conclusion: Instantons give absolute minima of the energy (= Yang-Mills) funct. $\mathcal{E}$.

Rk: For $G = SO(3)$ $(= PU(2))$ the above statement holds with

$$\mathcal{E}(A) \geq 2\pi^2 |p_1(Q)|$$

$$\begin{array}{ccc} U(2) & \xrightarrow{Ad} & Aut(su(2)) \\ \downarrow & \nearrow_{\cong} & \\ U(2)/U(1) & & \\ \cong & & \\ PU(2) & & \end{array}$$

and we have

$$\mathcal{E}(A) = \begin{cases} 8\pi^2\left(-\frac{1}{4}p_1(Q)\right) + 2\|F_A^+\|^2_{L^2} \\ -8\pi^2\left(-\frac{1}{4}p_1(Q)\right) + 2\|F_A^-\|^2_{L^2} \end{cases}$$

(uniform treatment with

Def^{n} (instanton number)

$$\kappa(P) = \begin{cases} \langle c_2(P), [X] \rangle & \text{for } G = U(n) \text{ or } SU(n) \\ -\frac{1}{4}\langle p_1(P), [X] \rangle & \text{if } G = SO(3) \end{cases}$$

Rk: we won't work with $G = U(n)$ because:

$$u(n) = su(2) \oplus i\mathbb{R}$$

Lie alg. splitting

Rk: If the $SO(3)$-bundle $Q \to X$ arises via the reduction

$$U(2) \xrightarrow{\rho} PU(2) = U(2)/\underbrace{Z U(2)}_{\text{centre}}$$

$$= U(2)/U(1)$$

$$\cong SO(3)$$

from a $U(2)$ (or $SU(2)$-) bundle $P \to X$

i.e. $Q = P \times_\lambda \mathfrak{su}(2)$.

Then

$$p_1(Q) = p_1(Q \times_{\mathfrak{so}} \mathbb{R}^3)$$

$$= p_1(P \times_{ad} \mathfrak{su}(2))$$

$$= -c_2(P \times_{ad} \mathfrak{sl}(2))$$

$$= -c_2((E \otimes E^*)_0)$$

$E = P_E \times_{\mathfrak{so}} \mathbb{C}^2$

$$= -4c_2(E) + c_1(E)^2$$

use Chern-character

<u>Exercise</u>: Show this using Chern-Weil theory and $\lambda_*: \mathfrak{u}(2) \to \mathfrak{su}(2)$.

Instantons on S^4

$P_+ = S^7 \subseteq \mathbb{H}^2$ ← before.

$$\begin{array}{c} P_+ = S^7 \subseteq \mathbb{H}^2 \\ \Big\downarrow {\scriptstyle \text{right } S^3\text{-action}} \\ S^4 = \mathbb{HP}^1 \\ (= \mathbb{H} \cup \{\infty\}) \end{array}$$

S^3 acts by isometries

$\Longrightarrow$ got a connection A by taking orthog. complement to S^3-action

Rk: A is not only $su(2) = S^3 = Sp(1)$-invariant, but also $Sp(2)$-invariant.

$\mathbb{H}$-lin. isom. of $\mathbb{H}^2$

We
Claim that A is ASD
(anti-self-dual)

We trivialise P_+ over $\mathbb{H} \subseteq \mathbb{HP}^1$ via the chart map

$$\begin{aligned} \mathbb{H} &\xrightarrow{f} \mathbb{HP}^1 \\ x &\longmapsto [x:1] \end{aligned}$$

$$\begin{tikzcd} f^* S^7 \arrow[r] \arrow[d] & S^7 \arrow[d] \\ \mathbb{H}^4 \arrow[r, "f"] \arrow[u, bend left, "s"] & f(\mathbb{H}) \subseteq \mathbb{HP}^1 \end{tikzcd}$$

We define the section

$$s(x) := \frac{(x, 1)}{(1+|x|^2)^{1/2}} \in S^7$$

Seen: $\omega_A = \operatorname{Im}(\bar{x}\,dx + \bar{y}\,dy)$

$= -\operatorname{Im}(d\bar{x}\cdot x + d\bar{y}\cdot y)$

(Exercise)

Connection 1-form is

$$a := s^*\omega_A$$

$$= \frac{\operatorname{Im}(\bar{x}\,dx)}{1+|x|^2}$$

$$s^*\Omega_A = da + \frac{1}{2}[a\wedge a]$$

$$= \dots = \operatorname{Im}\left(\frac{d\bar{x}\wedge dx}{(1+|x|^2)^2}\right)$$

$$x = x_0 + i x_1 + j x_2 + k x_3$$

$\in$ quaternions

$$\begin{cases} i^2 = -1 \\ ij = k \end{cases}$$

$\Rightarrow$

$$d\bar{x} \wedge dx = 2i\,(dx^0 \wedge dx^1 - dx^2 \wedge dx^3) + 2j\,(dx^0 \wedge dx^2 - dx^3 \wedge dx^1) + 2k\,(dx^0 \wedge dx^3 - dx^1 \wedge dx^2)$$

This is anti-selfdual w.r.t. $(\mathbb{H}^4, \text{eucl.})$

<u>Outlook</u>:

$$(\mathbb{H}, \text{eucl.}) \xrightarrow{f} (S^4, g_{\text{round}})$$

Claim: f is a conformal map

$$f^* g_{\text{round}} \stackrel{!}{=} t^2 \cdot g_{\text{Eucl}}$$