Instantons on S^4

$P_+ = S^7 \subseteq \mathbb{H}^2$ $\leftarrow$ before.

$\downarrow$ right S^3-action

$S^4 = \mathbb{HP}^1$
$(= \mathbb{H} \cup \{\infty\})$

S^3 acts by isometries $\Rightarrow$ got a connection A by taking orthog. complement to S^3-action

Rk: A is not only $SU(2) = S^3 = Sp(1)$-invariant, but also $Sp(2)$-invariant.

$\mathbb{H}$-lin. isom. of $\mathbb{H}^2$

We claim that A is ASD (anti-self-dual)

We trivialize P_+ over $\mathbb{H} \subseteq \mathbb{HP}^1$ via the chart map

$$\begin{aligned} \mathbb{H} &\xrightarrow{f} \mathbb{HP}^1 \\ x &\longmapsto [x:1] \end{aligned}$$

$$\begin{tikzcd} f^*S^7 \arrow[r] \arrow[d] & S^7 \arrow[d] \\ \mathbb{H}^4 \arrow[r, "f"] \arrow[u, bend left, "s"] & f(\mathbb{H}) \subseteq \mathbb{HP}^1 \end{tikzcd}$$

We define the section

$$s(x) := \frac{(x,1)}{(1+|x|^2)^{1/2}} \in S^7$$

Seen: $\omega_A = \operatorname{Im}(\bar{x}\,dx + \bar{y}\,dy)$

$= -\operatorname{Im}(d\bar{x}\cdot x + d\bar{y}\cdot y)$

(Exercise)

Connection 1-form is

$$a := s^*\omega_A$$

$$= \frac{\operatorname{Im}(\bar{x}\,dx)}{1+|x|^2}$$

$$s^*\Omega_A = da + \frac{1}{2}[a\wedge a]$$

$$= \dots = \operatorname{Im}\left(\frac{d\bar{x}\wedge dx}{(1+|x|^2)^2}\right)$$

$$x = x_0 + i x_1 + j x_2 + k x_3$$

$\in$ quaternions

$$\begin{cases} i^2 = -1 \\ ij = k \end{cases}$$

$\Rightarrow$

$$d\bar{x} \wedge dx = 2i\,(dx^0 \wedge dx^1 - dx^2 \wedge dx^3) + 2j\,(dx^0 \wedge dx^2 - dx^3 \wedge dx^1) + 2k\,(dx^0 \wedge dx^3 - dx^1 \wedge dx^2)$$

This is anti-selfdual wrt. $(\mathbb{H}^4, \text{eucl.})$

<u>Outlook</u>:

$$(\mathbb{H}, \text{eucl.}) \xrightarrow{f} (S^4, g_{\text{round}})$$

<u>Claim</u>: f is a conformal map

$$f^* g_{\text{round}} \overset{!}{=} t^2 \cdot g_{\text{Eucl}}$$

In exercise class:

$$\begin{array}{c} P_+ = S^7 \\ \downarrow \\ S^4 \end{array} \qquad \omega_{A_+} = \operatorname{Im}(\bar{x}dx + \bar{y}dy)$$

We verified: $\langle c_2(P), S^4 \rangle = 1$.

and: A_+ is anti-selfdual

up to the following result:

<u>Def</u>: A diffeom.

$$f : (M, g) \longrightarrow (N, h)$$

between Riem. mflds is **conformal**, if f^*h is conformally equiv. to g, which means $\exists$

$$t : M \longrightarrow \mathbb{R}^*_+ \quad \text{s.th.}$$

$$f^*h = t^2 \cdot g$$

Example: $z \mapsto \lambda \cdot z$ for some $\lambda > 0$

$$\begin{array}{ccc} \mathbb{R}^4 & \longrightarrow & \mathbb{R}^4 \\ \cap\!\!\mid & & \cap\!\!\mid \\ S^4 & \longrightarrow & S^4 \end{array}$$

chart from stereogr. proj.

extends to a conformal diffeo $f_\lambda : S^4 \longrightarrow S^4$

(the chart map is a conformal diffeo onto its image)

Prop^n: Suppose we have a bundle map

$$\begin{array}{ccc} P & \xrightarrow{\Phi} & P' \\ \downarrow & & \downarrow \\ (X^4, g) & \xrightarrow{f} & (X', g') \end{array}$$

Φ of type id (G-equiv.)

covering a conf. diffeom. f.
Let A' be a conn on P'.
Then A' is (anti) selfdual
$\iff A := \Phi^* A$ is (anti) selfdual

Pf: $A := \Phi^* A'$

i.e. $\omega_A = \Phi^* \omega_{A'}$

Φ gives $\bar{\Phi} : ad(P) \xrightarrow{\cong} ad(P')$

$[p, X] \mapsto [\Phi(p), X]$

fibrewise isom. covering f.

and

$$F_A = (\bar{\Phi})^{-1} F_{A'}$$

$\Omega^2(X, ad(P))$, more precisely

$$(F_A)_x(\xi, \eta) = (\bar{\Phi})_x^{-1} F_{A'}(df(\xi), df(\eta))$$

On „diff. form-part" this is f^*.

Now

$$(F_A)^{+_{f^*g'}} = 0 \iff F_A^{+_g} = 0 \quad (*)$$

$$(\bar{\Phi})^{-1} F_{A'}^{+_{g'}}$$

(*) uses that $*\big|_{\Lambda^2 T^* X^4}$

is conf. invariant $\square$

<u>Exercise:</u> $Sp(2) \curvearrowright S^7$ acting on fibrewise

$$\begin{array}{c} S^7 \\ \downarrow \\ S^4 \end{array}$$

A_f is invariant under this $Sp(2)$-action

$$Sp(2) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, g^* g = id \right\}$$

where $g^* = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$

group preserving

$$\left\langle \begin{pmatrix} z \\ w \end{pmatrix}, \begin{pmatrix} z' \\ w' \end{pmatrix} \right\rangle_{\mathbb{H}} = \bar{z} z' + \bar{w} w'.$$

In fact $Sp(2) \cong Spin(5)$

$$\begin{tikzcd} S^7 \arrow[r, "\text{acts by isom}"', "Sp(2)"'] \arrow[d] & S^7 \arrow[d] \\ S^4 \arrow[r, "SO(5)"', "\text{acts by isom.}"'] & S^4 \end{tikzcd}$$

gives

$$\begin{tikzcd} Sp(2) \arrow[d, "2:1"] \\ SO(5). \end{tikzcd}$$

$\boxed{\text{both have } \dim = 10}$

Thm $Conf(S^4) \cong SO(5,1)$

def. by isom. of a Lorentz metric on $\mathbb{R}^6 = \mathbb{R}^5 \oplus \mathbb{R}$

Furthermore, $Conf(S^4)$ is generated by $SO(5)$ and dilatations given by $z \mapsto \lambda \cdot z \quad \lambda > 0$

in stereogr. proj:

$$\dim(SO(5,1)) = 15$$

Another coincidence:

$$SL(2,\mathbb{H}) = \{ A \in M_2(\mathbb{H}) \mid \det A = 1 \}$$

$\boxed{\text{has dim. } 15}$

Dieudonné determinant (takes value in $\mathbb{H}/[\mathbb{H},\mathbb{H}] \cong \mathbb{R}$)

$$(*) \quad \begin{tikzcd} S^7 \arrow[r, "SL(2{,}\mathbb{H})"] \arrow[d] & S^7 \arrow[d] \\ S^4 \arrow[r, dashed, "SO(5{,}1)"] & S^4 \end{tikzcd}$$

$g \in SL(2,\mathbb{H})$ sends fibres to fibres and covers conformal maps of S^4

gives: $SL(2,\mathbb{H}) \xrightarrow{2:1} SO(5,1)$.

Analogy:

$$\begin{tikzcd}
S^3 \arrow[r, "\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})"] \arrow[d] & S^3 \arrow[d] \\
\mathbb{C}P^1 = S^2 = \mathbb{C} \cup \{\infty\} \arrow[r, "\text{Möbius-transf.}"] & S^2 \cong \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\} \\
\mathbb{C} \arrow[u, hook] \arrow[r] & \mathbb{C} \arrow[u, hook]
\end{tikzcd}$$

$$z \longmapsto \frac{az+b}{cz+d}$$

So $\mathrm{Conf}(S^2) \cong \mathrm{Bihol}(\mathbb{C}P^1)$

2:1-covered by $SL(2,\mathbb{C})$

$\circlearrowright S^3$

One can show

$$\mathrm{Stab}(A_f) = Sp(2)$$

(Exercise 2 above)

Via the above Prop^n and $(*)$ we get a family of ASD conn. on $P_f \to S^4$ given by

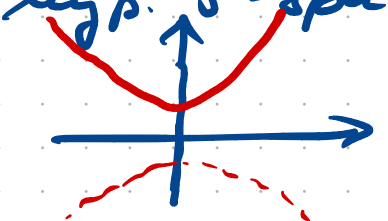
$$Sl(2,\mathbb{H}) / Sp(2) \cong SO(5,1) / SO(5)$$

$= \mathrm{Conf}(S^4)$

$$\cong H^5$$

5-dim'l hyperbolic space

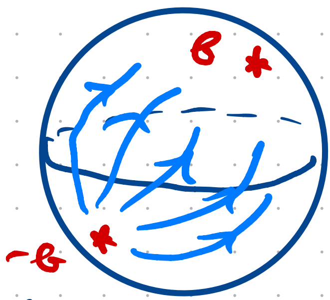
(hyperboloid model for hyp. 5-space



Next: Explicit descr. of a 5-dim'l family parametr. ASD instantons on $P_+ = S^7 \to S^4$:

Had $z \mapsto \lambda \cdot z$ $\quad \lambda \in \mathbb{R}_{>0}$

For $\{b, -b\}$ antip.-od. pts on S^4 we conjugate $z \mapsto \lambda \cdot z$ with an elt in $SO(5)$ mapping
$$b \mapsto N$$
$$-b \mapsto S$$



$\Rightarrow$ Get $\tau_{b,\lambda}$

We get a 5-dim'l family

$$C: \begin{cases} S^4 \times \mathbb{R}^*_{>0} \longrightarrow \mathrm{Conf}(S^4) \\ (b, \lambda) \longmapsto \tau_{b\lambda} \end{cases}$$

b = centre, λ = scale

Observe that $\tau_{b,\lambda} = \tau_{-b,\frac{1}{\lambda}}$.

$\Rightarrow C$ restrict/descends to

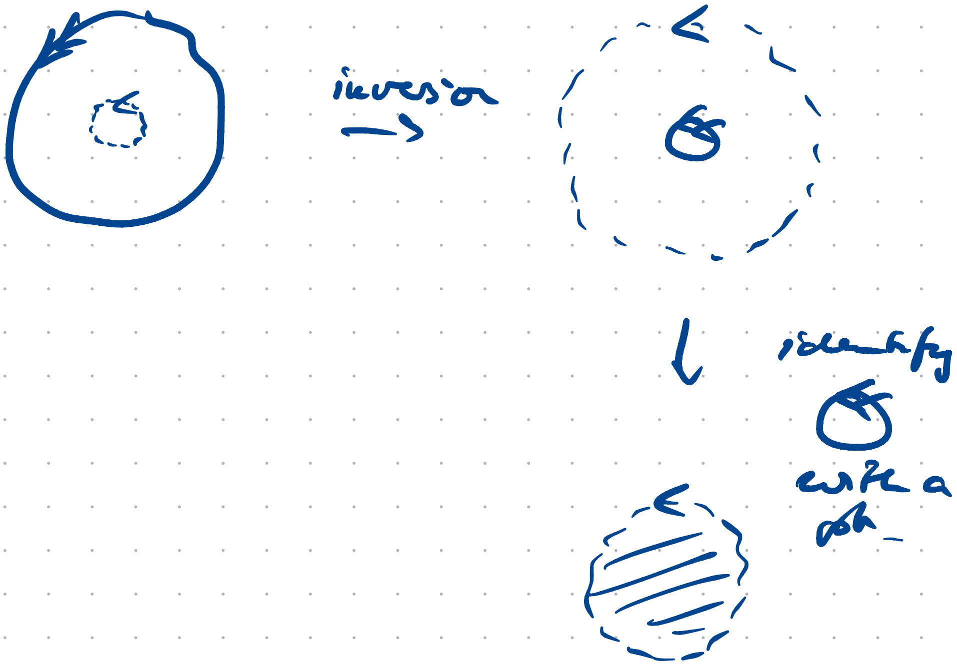
$$S^4 \times (0,1] \big/ \{1\} \times S^4 \longrightarrow \mathrm{Conf}(S^4)$$

$$\cong$$

$$B^5$$

$(\tau_{b,1} = \mathrm{id} \ \forall b)$

Picture:



Def^n $M_k(X^4) = \{A$ anti-self-dual conn. in $P \to X$ with $k(P) = k\} / \mathrm{Aut}(P)$

The previous map

$$S^4 \times (0,1] / \sim \;\longrightarrow \mathrm{Conf}(S^4)$$

gives rise to a $\mathring{B}^5$ worth of ASD conn. on S^4

Thm [ADHM: Atiyah - Hitchin - Drinfeld - Manin]

The space $M_1(S^4)$ is homeom to

$$\mathrm{Conf}(S^4)/SO(5) \cong SO(5,1)/SO(5) \cong Sp(2,\mathbb{H})/Sp(2) \cong \mathbb{H}^5$$

via the above construction.

$$[\lambda, b] \longmapsto \Phi^*_{\tau_{\lambda,b}} A_+$$
$$\rotatebox{90}{$\in$}$$
$$S^4 \times (0,1]/\sim$$

In particular, we get all instantons in this way from A_+.

<u>Rk</u>: [ADHM] have described all $\mathcal{M}_k(S^4)$ $k \geq 0$ by a generalising constr.

What happens in the local trivialisation

$$s(x) = \frac{(x,1)}{(1+|x|^2)^{1/2}}$$

$$\begin{tikzcd} S^7/_{\mathbb{H}} \arrow[r, hook] \arrow[d] & S^7 \arrow[d] \\ \mathbb{H} \arrow[u, bend left, "s"] \arrow[r, hook] & S^4 \end{tikzcd}$$

Then

$a := s^* \omega_{A_t}$

$$\begin{tikzcd} S^7 \arrow[r, "\Phi_{\tau_{\lambda,s}}"] \arrow[d] & S^7 \arrow[d] \\ S^4 \arrow[r, "\tau_{\lambda,s}"'] & S^4 \end{tikzcd}$$

$$s^*\left(\omega_{\Phi_{\tau_{\lambda,N}}^* A_t}\right)$$

$$= \tau_\lambda^*(a) \qquad \tau_\lambda : x \mapsto \lambda x$$

$$\tau_\lambda^* a = \frac{\lambda^2 \, \mathrm{Im}(\bar{x}dx)}{\lambda^2 \cancel{1} + \cancel{\lambda^2} |x|^2} = \frac{\mathrm{Im}(\bar{x}dx)}{\lambda^2 + |x|^2}$$

$$= \frac{\mathrm{Im}(\bar{x}dx)}{\frac{1}{\lambda^2} + |x|^2}$$

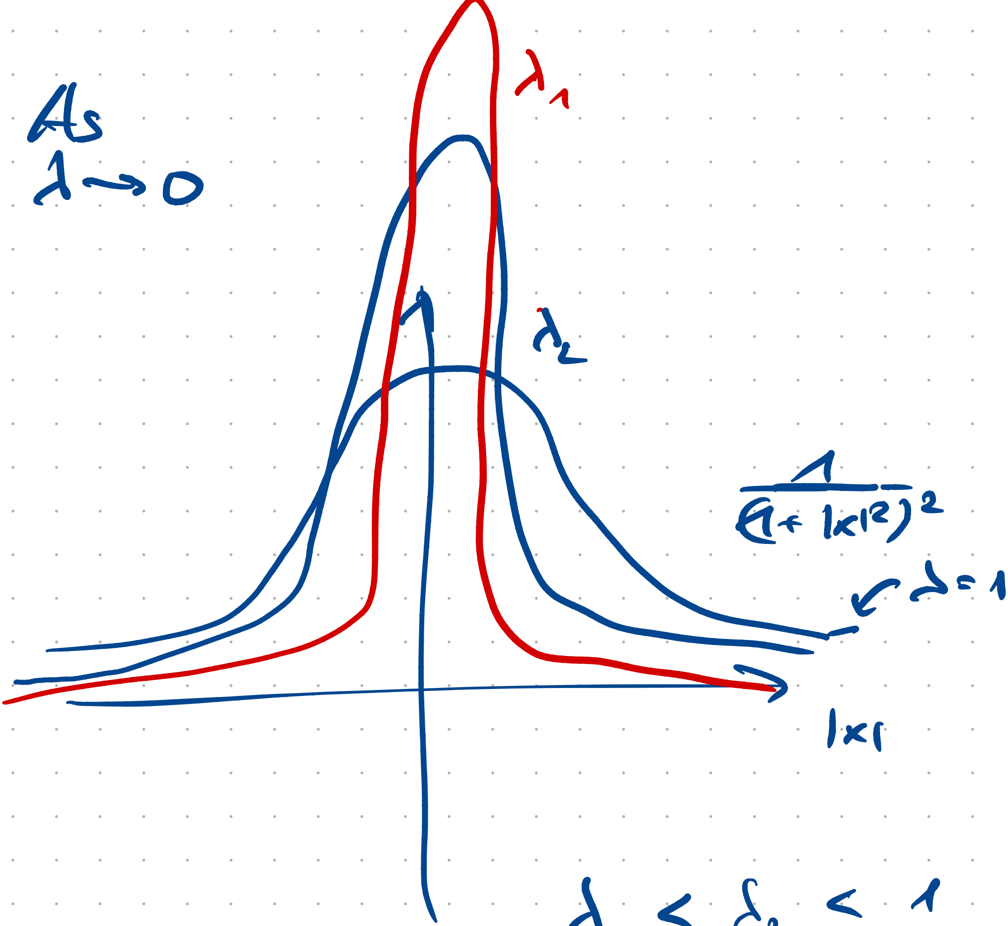
Correction $\qquad \to 0$ on $\mathbb{R}^4 \setminus \{0\}$ as $\lambda \to 0$

$$\tau_\lambda^*(s^* \Omega_{A_t}) = s^*\left(\Omega_{\Phi_{\tau_{\lambda,N}}^* A_t}\right)$$

$$= \tau_\lambda^*\left(a + \tfrac{1}{2}[a \wedge a]\right)$$

$$= \frac{\lambda^2 \, d\bar{x} \wedge dx}{(\cancel{1}\lambda^2 + \cancel{\lambda^2}|x|^2)^2} = \frac{\lambda^2 \, d\bar{x} \wedge dx}{(\lambda^2 + |x|^2)^2}$$

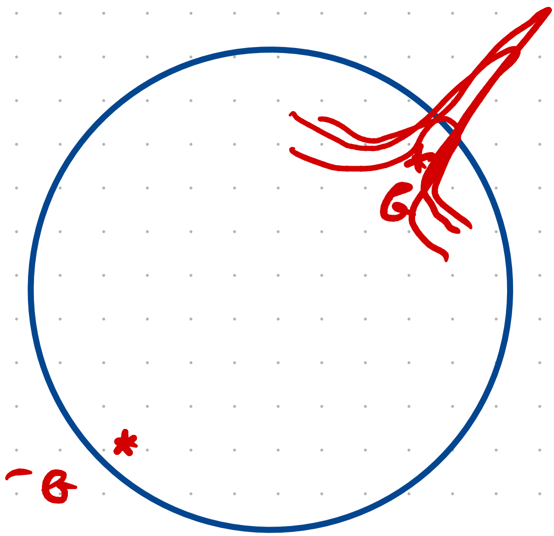
As
$\lambda \to 0$



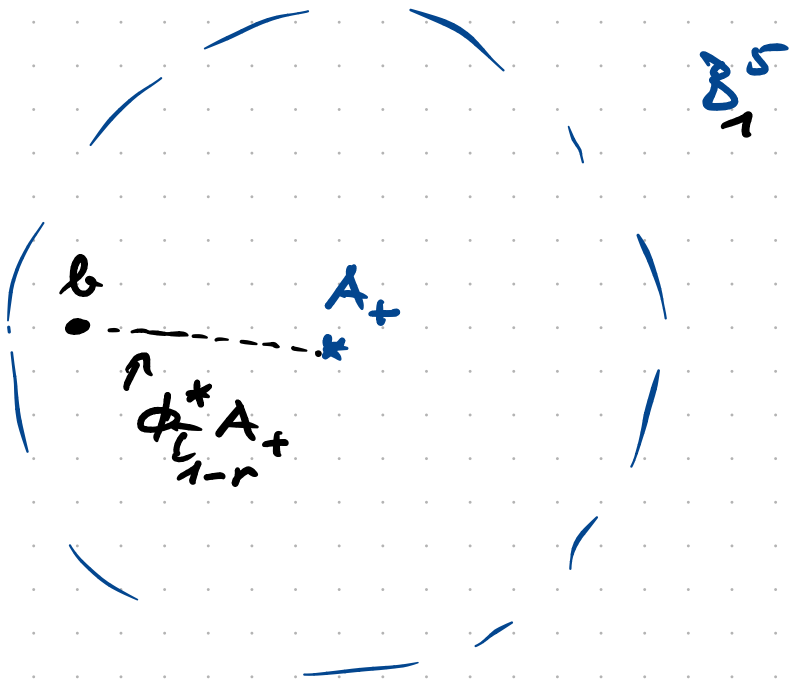
$\lambda_1 < \lambda_2 < 1$

$\Rightarrow$ get a concentration of curvature at 0 as $\lambda \to 0$.

Because of the shape of these curvatures for diff. λ and concentr. of centres $b \in S^4$, we see that in the above family no two are gauge equivalent.



Picture for $\mathcal{M}_1(S^4)$:



r = radial.

Seems

Natural: $S^4 = \partial B_1^5$

as a „limit" and

so

$$\overline{\mathcal{M}_1(S^4)} = \overline{B_1^5}$$

seems to have a compactification.

Exercise: For $\lambda = 0$

$$\tau_\lambda^* a = \frac{\operatorname{Im}(\bar{x}\, dx)}{|x|^2}$$

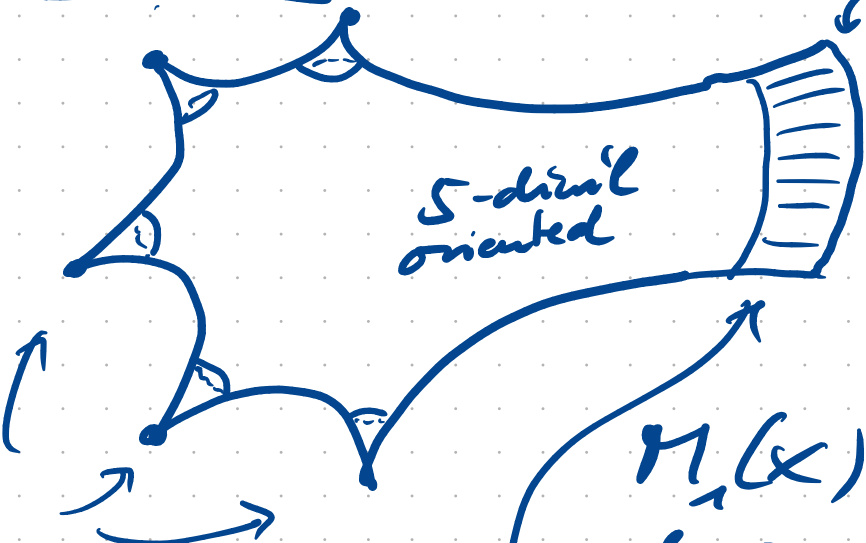
on $\mathbb{R}^4 \setminus \{0\}$

is gauge-equivalent to the trivial (flat) connection.

Outline of proof of Donaldson's theorem

Thm: X neg. def. smooth, $\pi_1(X)=1$. Then $Q_X = \begin{pmatrix} -1 & & \\ & \ddots & \\ & & -1 \end{pmatrix}$

Pf sketch:



$\partial = X^4$ as pts of concentration

$M_1(X)$
charge $= 1$ - moduli sp

„Taubes' collar" $\cong (0,1] \times X^4$

singularities which are $\operatorname{cone}(\overline{\mathbb{CP}^2}) \cong (B_1\mathbb{C}^3)/S^1$

norm-1-ball in $\mathbb{C}^3$

by cpx. multipl.

Arg:

$$k := \#\left\{ \{c,-c\} \in H^2(X;\mathbb{Z}) \;\middle|\; \langle c^2, [X] \rangle = -1 \right\}$$

$$\Rightarrow \quad Q_X = \left(\begin{array}{c|c} \underbrace{\begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix}}_{k} \Big\}k & \\ \hline & Q' \end{array} \right)$$

Need to show: $\boxed{k = b_2(X)}$

$(\Rightarrow$ no $Q')$

By analysis of singularities we see that

$k = \#$ cones on $\overline{\mathbb{CP}^2}$

Fact $\operatorname{sign}(Q_X) = b_2^+(X) - b_2^-(X)$
$= -b_2(X)$ for X neg. def.

is invariant under oriented 5-dim'l cobordism

But

$$\mathrm{sign}\Big(\underset{k}{\coprod} \overline{\mathbb{CP}}^2\Big) = -k$$

$$\overset{M_1(X)}{\underset{\text{provides ancor. coord}}{\Longrightarrow}} \quad k = b_2(X).$$

$\boxed{/\!\!/s}$

That was dessert.
Have to do:

* local analysis:
  $M_k(X)$ is locally a mfld of fin. dim.
  $\dim M_k(X) = 8k + 3(b_1 - b_2^+ - 1)$

  $\rightsquigarrow$ need to work with Banach spaces, implicit fctn thm!

* global analysis:
  understand compactification, and "inverses" like Tauber' collar

* orientations

* analysis of Kingelandsches ASD
  = connections with non-trivial stabiliser under action of Aut(P).

No class on Thursday 3 June