Reducibles

$$M_k(X) = \{ A \text{ conn on } P \text{ with } k(P) = k \mid F_A^+ = 0 \} / \mathcal{G}$$

gauge group $\rightarrow$ $\mathcal{G} = \mathrm{Aut}(P)$

The asd equ is in fact gauge invariant.

If $u \in \mathcal{G}$, then

$$\boxed{F_{u^*(A)} = \mathrm{ad}_{\bar{\varphi}_u} \circ F_A}$$

(here

$$\mathrm{Aut}(P) \cong \{ \varphi : P \to G \ \ \mathrm{Ad}\text{-equiv.} \}$$
$$u \rightsquigarrow \varphi_u$$

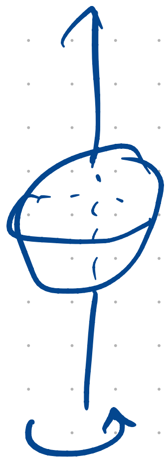
$$\cong \Gamma(M; P \times_{\mathrm{Ad}} G) \ni \bar{\varphi}_u$$

$$\Rightarrow \quad F_A^+ = 0 \iff F_{u(A)}^+ = 0$$

$\mathcal{A}_P := \{A \text{ conn on } P\}$

$\mathcal{G} \curvearrowright \mathcal{A}$

$$M_k(X) \subseteq \mathcal{A}/\mathcal{G}$$



S^1-action
quotient

<u>Def</u>: $\mathrm{Stab}_A := \{u \in \mathcal{G} \mid u(A) = A\}$

What can Stab_A be?

<u>First observation</u>: (for any Lie group G)

$$Z(G) = \{h \in G \mid hg = gh \text{ for all } g \in G\}$$

For $c \in Z(G)$, define

$$u_c(p) := p \cdot c$$

Then $u_c \in \mathcal{G}$:

$$u_c(pg) = pg\cdot c$$

$g \in G$

$$= p\cdot c g = u_c(p) g$$

$c \in Z(G)$

<u>Claim:</u> $u_c^* A = A$

Pf: Seen

$$\omega_{u^*A} = \omega_A + \varphi_u^{-1} d_A \varphi_u$$

where $\varphi_u \in \Omega^0_{Ad}(P;G)$

(G-equiv.

In our situation $\varphi_u \equiv c$

Therefore $d_A \varphi_u \equiv 0$

$$\Rightarrow \omega_{u^*A} = \omega_A$$ $\blacksquare$

So if $Z(G) \neq \{e\}$, then

$\mathrm{Stab}_A \neq \{e\}$

But this is not a problem,
may factor the action

$$\mathfrak{g} \curvearrowright \mathcal{A} \quad \text{via}$$

$$\mathfrak{g}/_{Z(G)} \curvearrowright \mathcal{A}.$$

<u>Remind:</u>

$$\left\{ u \in \mathcal{G} \to \varphi_u(p) := \text{the unique } g \in G \text{ s.th. } p\,\varphi_u(p) = u(p). \right.$$

From

$$\omega_{u^*A} = \omega_A + \varphi_u^{-1} d_A \varphi_u$$

we see that $u^*A = A$

$$\Rightarrow \boxed{d_A \varphi_u \equiv 0}$$

That means φ_u is determined everywhere from $\varphi_u(p)$ if M is connected

$\left(\varphi_{u}|_{\pi^{-1}(x)} \right.$ def. by $\varphi_u(p)$ for $p \in \pi^{-1}(x)$

by equivariance,

elsewhere from parallel transport)

In that sense (*)

$$\boxed{\begin{aligned} \mathrm{Stab}_A &\subseteq G \\ \varphi &\longmapsto \varphi(p) \end{aligned}}$$

<u>Prop</u>":

$$(\mathrm{Stab}(A))_p = \mathrm{Centr}(\mathrm{Hol}_p(A))$$

considered in (*) sense

<u>Def</u>: $H \subseteq G$, $\mathrm{Centr}(H) = \{ g \in G \mid gh = hg \ \forall h \in H \}$

"centralizer" or "commutant".

$$\mathrm{Hol}_p(A) = \{ g \in G \mid pg = \tilde{\gamma}_p(1),$$

where $\tilde{\gamma}_p$ is the
A-parallel lift of
some curve
$\gamma : [0,1] \to M$,
where $\tilde{\gamma}_p$ starts at $p \}$

<u>Proof</u>: Suppose $u^* A = A$

$\Rightarrow d_A \varphi_u \equiv 0$

<u>Claim</u>:

$\varphi(pg) = \varphi(p)$ for all $g \in \mathrm{Hol}_p(A)$.

$pg = \tilde{\gamma}_p(1)$ $\quad$ $\tilde{\gamma}_p$ some A-par. path in P, st. at p.

$$\frac{d}{dt}(\varphi \circ \tilde{\gamma}_p)(t) = d\varphi(\dot{\tilde{\gamma}}_p(t)) = d_A \varphi(\dot{\tilde{\gamma}}_p(t)) = 0$$

$$\Longrightarrow \quad \varphi(\tilde{\gamma}_p(0)) = \varphi(\tilde{\gamma}_p(1))$$

$$\varphi(p) \qquad\qquad \varphi(pg).$$

Otoh $\quad \varphi(pg) = g^{-1}\varphi(p)g$

$$\Rightarrow \quad \varphi(p) = \varphi(pg) = g^{-1}\varphi(p)g$$

$$\Rightarrow \quad g\,\varphi(p) = \varphi(p)\,g$$

$$\Longrightarrow \quad \varphi(p) \in \mathrm{Centr}(\mathrm{Hol}_p(A)).$$

(⊇) From $h \in \mathrm{Centr}(\mathrm{Hol}_p(A))$ one constructs a G-equiv. map $\varphi: P \to G$:

$$\varphi(q) := h$$

if $q = \tilde{\gamma}_p(1)$ where $\tilde{\gamma}_p$ is a A-parallel path (not nec. closed) starting at p

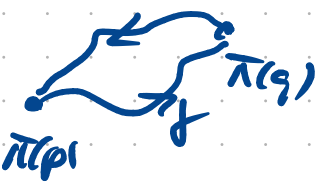
$\varphi|_{\pi^{-1}(\pi(p))}$ defined by equivariance

$\tilde\sigma$ A-parallel from q' to $\pi^{-1}(\pi(p))$

$$q' =: qk$$

Certainly this results in

$$d_A\varphi \equiv 0$$



Do we have $\varphi(qk) = k\,\varphi(q)\,k^{-1}$ $\forall k \in G$?

Now

$R_{k^{-1}}\tilde\sigma$ starts at q and ends

is A-parallel at $\tilde\gamma_p * R_{k^{-1}}\sigma(1)$.

$\pi\circ(\gamma_p * R_{t\to\sigma})$ is a closed loop.

$\Longrightarrow (\gamma_p * R_{t\to\sigma})(1) = pg$

for some

$g \in Hol_p(A)$.

$\varphi(g') = \varphi(\underset{\underset{\pi^{-1}(\pi(p))}{\Uparrow}}{\tilde{\sigma}(1)})$

.....

$\blacksquare$

So **stabilisers** of $\mathcal{O}_J$ correspond to $\mathrm{Centr}(\underline{\mathrm{Hol}_p(A)})$

subgp. of G

Not.: $C(H \leq G)$
$:= \mathrm{Centr}(H \leq G)$

Prop:

* $C(kHk^{-1}) = k\, C(H)\, k^{-1}$

* $K \leq H \Longrightarrow C(H) \leq C(K)$

* $C(Z(G)) = G$

$C(\{e\}) = G$

Pf: Exercise!

$$\begin{array}{ccc} S(H) & & \\ \| & & \\ SU(2) & \xrightarrow[2:1]{\mathrm{ad}} & \mathfrak{so}(\mathfrak{su}(2)) \\ & & \| \\ & & \mathbb{R}^3 \end{array}$$

Examples:

$G = SU(2)$

What are the possible centralizers? of subgps $H \subseteq SU(2)$?

- $H = ZSU(2) = \{\pm 1\}$
  $\Rightarrow C(H) = SU(2)$

- $H = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \,\middle|\, z \in U(1) \right\}$
  $\cong U(1)$
  $\Rightarrow C(H) = H$

  Same for discrete subgroups $K \subseteq U(1)$, unless K is central

- $H = Pin(2)$
  $= \left\{ \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \,\middle|\, z \in U(1) \right\}$
  $\cup \left\{ \begin{pmatrix} 0 & w \\ -\bar{w} & 0 \end{pmatrix} \,\middle|\, w \in U(1) \right\}$

Then $C(H) = ZSU(2)$

- $H = SU(2) \Rightarrow C(H) = Z_{SU(2)}$

Centraliser graph: $\longrightarrow$ (= inclusion)

$$\{\pm 1\} \longrightarrow U(2) \longrightarrow SU(2)$$

<u>Example:</u> $G = SO(3)$

- $C(SO(3)) = \{e\}$
- $\mathbb{Z}/2 \cong \left\{ \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathrm{id} \right\}$

$$C(\mathbb{Z}/2) = \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & \det A \end{array} \right) \middle| A \in O(2) \right\}$$

$$\cong O(2)$$

- $K_4 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$

$$= \left\{ A = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} \middle| \begin{array}{c} \det A \\ = 1 \end{array} \right\}$$

= sym. gp of cube

$\Rightarrow C(K_4) = K_4$

- $K = SO(2)$

$\Rightarrow C(K) = SO(2)$

- $C(O(2)) \cong \mathbb{Z}/2$

<u>Def^n</u>: A connection is called <u>reducible</u> if $Stab(A) \neq Z(G)$.

$\rightsquigarrow$ singularities in $\mathcal{A}/\mathcal{G}$.

So for $SU(2)$-bundles the possible centralisers are

$$\{ Z SU(2) = \{\pm id\},\ U(1),\ SU(2) \}$$

stabilises of a generic connection

stabilises a connection leaving invariant $E = L \oplus L^{-1}$

stabilises the trivial connection

Sobolev completions

E hermitian/ Euclid. bundles

$\downarrow$

M oriented Riem.

Suppose ∇_A is a connection on E

For $s \in \Gamma(E)$, $p \geq 1$

$$\|s\|^p_{L^p_{k,\nabla_A}} := \sum_{\ell=0}^{k} \int_M |\nabla_A^{(\ell)} s|^p \, \mathrm{vol}_M$$

this is

$\nabla^{(1)}_A s = \nabla_A s \in \Gamma(T^*M \otimes E)$

$\nabla_A^{(2)}$ is $\nabla_{LC} \otimes \nabla_A$

(conn. on T^*M)

$\vdots$

$\nabla_A^{(\ell)}$ is $\nabla_{LC}^{\otimes \ell-1} \otimes \nabla_A$

$| \; |^p$ from pointwise inner product on $T^*M^{\otimes \ell-1} \otimes E$

Def^n:

$$L^p_k(M;E) = \overline{C^\infty(M;E)}^{\|\cdot\|_{L^p_{k,A}}}$$

Exercise:

If A and A' are smooth connections, then

$$\|\cdot\|_{L^p_{k,A}} \quad \text{and} \quad \|\cdot\|_{L^p_{k,A'}}$$

define equivalent norms.

Thm (Sobolev's embedding theorem)

In $\dim M = n$ we have inclusions

(1.) $L^p_k \hookrightarrow L^q_\ell$

if $k > \ell$ and the weights $w(L^p_k) = k - \frac{n}{p}$ satisfy

$$w(L^p_k) \geq w(L^q_\ell)$$

(2.) $L^p_k \hookrightarrow C^\ell$

if $w(L^p_k) > \ell$

In particular

$$L^p_k \hookrightarrow C^0 \quad \text{if } w(L^p_k) > 0$$

Examples:

On $\dim M = 4$ get

$$L^2_1 \hookrightarrow L^4, \quad L^p_1 \hookrightarrow C^0 \text{ if } p > 4$$

Thm (Rellich Lemma)

On a compact mfld (or a bounded domain $\subseteq \mathbb{R}^n$) the first inclusion is compact if the inequality between weights is strict.

Def: A lin. map is compact if it maps bounded sets to precompact sets (closure is compact)

No proofs!
[Gilbarg - Trudinger]

To motivate the Rellich lemma:

<u>Arzelà-Ascoli thm</u>:

X compact

$(f_n)_{n\in\mathbb{N}}$ a seq. $f_n : X \to \mathbb{R}$ or $\mathbb{C}$

s.th.

* (f_n) is equicontinuous
  (at every pt $x \in X$, the same ε-δ-arg. works <u>for all</u> n)

* $(f_n(x))$ is bounded $\forall x \in X$.

<u>Then</u> $\exists$ convergent subsequence of (f_n) in the C^0-top.

<u>Cor</u>: If $(f_n) \subseteq C^1(X)$ is $^{\text{pointwise}}_{\text{bounded}}$ and has <u>uniform bound on derivative</u>, then $\exists$ C^0-convergent subsequence. In particular

$C^1(X) \hookrightarrow C^0(X)$ is compact.