Exercise:

$$\mathbb{R}^n \longrightarrow \mathbb{R}^n$$

r_t : scaling by $t > 0$

$$x \longmapsto tx$$

Then $\varpi(L^p_k) = k - \frac{n}{p}$ is the exponent in t under which the highest order term $\|\nabla^k f\|_{L^p}$ scales:

$$\|\nabla^k r_t^* f\|_{L^p}$$

$$= t^k \|r_t^* \nabla^k f\|_{L^p}$$

$$= t^k \Big(\int_{\mathbb{R}^n} (r_t^* |\nabla^k f|)^p dx\Big)^{1/p}$$

var. tr.
$y = tx$
$dy = t dx$

$$= t^k (t^{-n})^{1/p} \|\nabla^k f\|_{L^p}$$

$$= t^{\varpi(k,p)} \|\nabla^k f\|_{L^p}.$$

Rk ⚠

On a mfld with fin. volume we have

$$L^p \hookrightarrow L^q \quad \text{if } q > p \geq 1$$

$\exists\, C > 0$ s.th.

$$\|f\|_{L^p} \leq C \cdot \|f\|_{L^q}$$

notice: if $|f(x)| \geq 1$ then $|f(x)|^q \geq |f(x)|^p$

But on non fin. volume spaces this is not true

e.g. $f : [1, \infty) \to \mathbb{R}_+ \; ; \; x \mapsto \frac{1}{x}$

Then $f \in L^2$, but
$f \notin L^1$.

We will prove some special cases of Sobolev embedding thm.

Prop: If $w(L_1^p) = 1 - \frac{n}{p} > 0$ then on a compact mfld (or fin. volume mfld)

$$L_1^p \hookrightarrow C^0$$

Need

Lemma: For $f \in C_0^1(\mathbb{R}^n)$ we have

compact support

$$|f(0)| \leq \int_{\mathbb{R}^n} \frac{1}{|x|^{n-1}} |\nabla f| \, dx$$

Lebesgue measure

Proof: Integrate along rays:

$$\int_0^\infty \frac{\partial f}{\partial r}(0 + \underline{e} r) \, dr = -f(0)$$

$\underline{e} \in S^{n-1}$

$$\Rightarrow \quad f(0) = -\int_0^R \frac{\partial f}{\partial r}(0+\underline{e}r)\,dr$$

$$\boxed{\begin{array}{l}\operatorname{supp} f\\ \subseteq B_R(0)\end{array}} \qquad = -\int_0^R (\nabla f \cdot e)(r\underline{e})\,dr$$

Hence,

$$|f(0)| = \frac{1}{\operatorname{vol}(S^{n-1})} \int_0^R \int_{S^{n-1}} |\nabla f \cdot e| \, dr\, d\omega$$

volume form on S^{n-1}

$$\leq \frac{1}{\operatorname{vol}(S^{n-1})} \int_0^R \int_{S^{n-1}} \frac{|\nabla f|}{r^{n-1}} \underbrace{\cdot\, r^{n-1}\,dr\,d\omega}_{= dx}$$

$$\leq \int_{B_R^n(0)} \frac{|\nabla f|}{|x|^{n-1}}\,dx \qquad \blacksquare$$

<u>Pf of Prop:</u>

Recall Hölder ineq:

If $\frac{1}{p} + \frac{1}{q} = 1$ then

$(p, q > 1)$

$$\underbrace{\int |fg|\,dx}_{\|fg\|_{L^1}} \leq \|f\|_{L^p} \|g\|_{L^q}$$

From Lemma:

$$|f(0)| \overset{\text{Hölder}}{\leq} \underbrace{\left(\int_{B_R^n(0)} \left(\frac{1}{|x|^{n-1}} \right)^q dx \right)^{1/q}}_{=: I} \cdot \|\nabla f\|_{L^p}$$

$$\left\{ \begin{aligned} &\frac{1}{p} + \frac{1}{q} = 1 \\ &\Rightarrow q = \frac{p}{p-1} \end{aligned} \right\}$$

I has to be finite!

For integrability of I at 0:

Need

$$\int_0^R \left(\frac{1}{r^{n-1}} \right)^q \cdot r^{n-1} dr < \infty$$

$$= \int_0^R r^{n-1-(n-1)\frac{p}{p-1}} dr$$

which is finite iff exp > -1

$$\Leftrightarrow \quad n-1-(n-1)\frac{p}{p-1} > -1$$

$$\Leftrightarrow \quad n > (n-1)\frac{p}{p-1}$$

$$\Leftrightarrow \quad \frac{n}{n-1} > \frac{p}{p-1}$$

$$\overset{(*)}{\Leftrightarrow} \quad p > n$$

(*) because
$$x \longmapsto \frac{x}{x-1} = 1 + \frac{1}{x-1}$$
is decreasing for $x > 1$

We have shown:

For $f \in C_0^\infty(\mathbb{R}^n)$
$$\|f\|_\infty \leq C(n,p) \cdot \|\nabla f\|_{L^p} \cdot \underbrace{\operatorname{diam}(\operatorname{supp} f)^{c(n,p)}}_{\to \infty}$$
as $\operatorname{diam} \to \infty$.

On a compact mfd M we get from this:

(**) $$\|f\|_\infty \leq C(n,p,M) \, \|f\|_{L_1^p(M)}$$
for $f \in C^1(M)$.

$\Rightarrow$ $\|\cdot\|_\infty$ is controlled by L_1^p-norm.

$$\begin{tikzcd}
 & L^\infty \\
\mathcal{C}^\infty \arrow[ur, hook] \arrow[dr, hook] & \\
 & L^p_1 \arrow[uu, dashed, "(**)"']
\end{tikzcd}$$

commutes

(**) $\leftarrow$ bded lin map

by passing to the limit in (**).

But really:

$$\begin{tikzcd}
\mathcal{C}^0 \arrow[r, hook, "\text{closed subspace}"] & L^\infty \\
\mathcal{C}^\infty \arrow[u, hook] \arrow[r, hook] & L^p_1 \arrow[u, dashed] \arrow[ul, dashed]
\end{tikzcd}$$

(If $(f_n) \subseteq \mathcal{C}^\infty$ is a Cauchy-sequence in $L^p_1(\Omega)$, then (**) gives a Cauchy seq. in C^0, so the limit is in C^0.)

We will consider:

$\mathcal{A}^p_k = L^p_k$ - completions of $\mathcal{A}$

$\uparrow$ sp. of conn.

affine over $\Omega^1(M; \mathrm{ad}(P))$

$\mathcal{G}^p_{k+1} = L^p_{k+1}$ - completions of gauge tr.

We want curvature

$$F_{A_0+a} = F_{A_0} + d_{A_0} a + \frac{1}{2}[a \wedge a]$$

<u>Remark</u>: Derivatives D of order $n \geq 0$ define <u>bounded</u> linear maps

$$L^p_k \xrightarrow{D} L^p_{k-n}$$

$\mathcal{G} \curvearrowright \mathcal{A}$
$(u, A) \mapsto A + u\, d_A u^{-1}$

Need some multiplication results.

„Borderline": $w(L^p_k) = 0$

<u>Thm</u> (multipl below borderline)

If $w(L^p_k) < 0$, $w(L^q_\ell) < 0$ then we have bounded multiplication

$$L^p_k \times L^q_\ell \hookrightarrow L^r_m$$

if $m \le \min(k, \ell)$

$$w(L^r_m) \le w(L^p_k) + w(L^q_\ell)$$

Rk: For $u, v < 1$ and $u+v<1$

$$\|fg\|_{L^{\frac{1}{u+v}}} \leq \|f\|_{L^{\frac{1}{u}}} \cdot \|g\|_{L^{\frac{1}{v}}}$$

follows from / is the Hölder inequality

Recall: $L^p_k \hookrightarrow L^r \quad \forall r \geq 1$

from Sobolev emb. thm.

if $w(L^p_k) = 0$

(but not $\hookrightarrow L^\infty$, for instance

$f: (0,1) \to \mathbb{R}$

$x \mapsto \log x$

$f \in L^r \; \forall r$, but unbounded)

and

$$L^p_k \hookrightarrow L^{\frac{1}{-\frac{w(k,p)}{n}}} = L^{\frac{1}{k/n - 1/p}}$$

(**) if $w(L^p_k) < 0$

(but $k/n - 1/p < 1$)

For $\|fg\|_{L^r_m}$ we need to control L^r-norms of

$\|\nabla^c(fg)\|_{L^r}$

for $c \leq m$

Notice: $\nabla^c(fg) = \sum M(c,a,b)\,(\nabla^a f)(\nabla^b g)$

(Leibniz rule)

$\nabla^a f \in L^p_{k-a} \hookrightarrow L^{\frac{1}{\frac{1}{p}-\frac{(k-a)}{n}}}$

$\nabla^b g \in L^q_{l-b} \hookrightarrow L^{\frac{1}{\frac{1}{q}-\frac{(l-b)}{n}}}$

$\xrightarrow{\text{Hölder ineq.}}$

$(\nabla^a f)(\nabla^b g) \in L^{\frac{1}{\frac{1}{p}-\frac{(k-a)}{n}+\frac{1}{q}-\frac{(l-b)}{n}}}$

$\hookrightarrow L^r$

because $w(L^r) = -\frac{n}{r}$

and we have

$$w\left(L^{\frac{1}{-w(k-a,p)-w(l-b,q)}}\right)$$

$$= +w(k-a,p) + w(l-b,q)$$

$$= w(k,p) + w(l,q) - a + b$$

$$\geqslant w(k,p) + w(l,q) - m$$

$$\geqslant w(L^r)$$

$\blacksquare$

Thm (Multiplication above the borderline)

If $w(L^p_k) > 0$ and

$$L^p_k \hookrightarrow L^q_\ell$$

($k \geq \ell$, $w(L^p_k) \geq w(L^q_\ell)$)

then we get

$$L^p_k \times L^q_\ell \longrightarrow L^q_\ell$$

i.e. L^q_ℓ is a module over L^p_k

and L^p_k is a Banach algebra

No proof but similar to above

<u>Thm</u> (Sobolev multipl. of the borderline)

$w(L^p_k) = 0 \qquad L^p_k \hookrightarrow L^q_\ell$

$(k \geq \ell$ and $w(L^q_\ell) \leq 0)$

Then we have bounded multiplication map

$$(L^p_k \cap L^\infty) \times (L^q_\ell \cap L^\infty) \longrightarrow (L^q_\ell \cap L^\infty)$$

If $w(L^q_\ell) < 0$, then we get

$$(L^p_k \cap L^\infty) \times L^q_\ell \longrightarrow L^q_\ell .$$

Illustration of Dilemma:

$$\int |\nabla f g|^q \mathrm{vol} \leq \int |\nabla f|^q |g|^q_{\mathrm{vol}} + \int |f|^q |\nabla g|^q_{\mathrm{vol}}$$

If we want to use Hölder ineq.

$$\frac{1}{u} + \frac{1}{v} = 1$$

Pf (of borderline case)

Extreme case

$$\| f (\nabla^e g) \|_{L^q} \leq \| f \|_\infty \| g \|_{L^q_e}$$

Now assume $a > 0$:

$$\| \underbrace{(\nabla^a f)}_{\in L^p_{k-a}} (\nabla^b g) \|_{L^q}$$

$\in L^p_{k-a} \leftarrow$ weight < 0.

To use the Hölder ineq. we need

$$\frac{1}{q} \overset{!}{\geq} -\frac{w(k-a,p)}{n} - \frac{w(l-b,q)}{n} \quad \text{by } (**)$$

$$= -\frac{(l-b-a)}{n} + \frac{1}{q}$$

and this holds because $b+a \leq l$

So by Hölder ineq.

$$\|(\nabla^a f)(\nabla^b g)\|_{L^q}$$

$$\leq \|(\nabla^a f)(\nabla^b g)\|_{L^{\frac{1}{-\frac{w(l-b,q)}{n}-\frac{w(k-a,p)}{n}}}}$$

Hölder

$$\leq \|\nabla^a f\|_{L^{\frac{1}{-\frac{w(k-a,p)}{n}}}} \cdot \|\nabla^b g\|_{L^{\frac{1}{-\frac{w(l-b,q)}{n}}}}$$

$$\leq \|f\|_{L^p_k} \cdot \|g\|_{L^q_\ell}$$

works unless

$w(\ell-k, q) = 0$

$\iff k = 0$ and $w(\ell, q) = 0$

In that case where it fails

$$\|\nabla^q f\, g\|_{L^q} \leq \|\nabla^q f\|_{L^q} \cdot \|g\|_\infty$$

$\blacksquare$