Ex. 1: a) Claim: Two smooth vector bdles over a fin. dim. smooth mfd M are isom. as bdles iff their $C^\infty(M)$-modules are isom.

Fact: Every smooth vector bdle over M has a complement, i.e. a vector bdle s.t. the Whitney sum of the two is trivial.

Step 1: $\Gamma(E)$ is a free C^∞-module of rank n iff E is trivial with fibre $\mathbb{R}^n$.

Pf: "E trivial $\Rightarrow$ $\Gamma(E)$ free":

Any global frame $e_1, \dots, e_n$ of $E \to M$ induces the C^∞-iso

$$\Gamma(E) \xrightarrow{\cong} C^\infty(M)^n,$$

$$\sum_{i=1}^n f_i e_i \longmapsto (f_1, \dots, f_n)$$

"$\Gamma(E)$ free $\Rightarrow$ E trivial":

Let $e_1, \dots, e_n \in \Gamma(E)$ be a C^∞-basis.

Then for every $m \in M$, $e_1(m), \dots, e_n(m)$ constitute a basis of E_m:

Assume $e \in E_m$ is not in $\operatorname{span}\{e_1(m), \dots, e_n(m)\}$

Then let U be a nbhd of m on which E is trivial & let $\beta : M \to \mathbb{R}$ be a smooth fct. w/ $\beta(m) = 1$, $\operatorname{supp}\beta \subseteq U$.

Then $m \longmapsto \beta(m) \cdot e \in \Gamma(E)$ is a section that cannot be written as a lin. comb. of $e_1, \dots, e_n$ ↯

Now let E, E' are arbitrary v.b over M.
If $f: E \to E'$ is an iso, then $f^*: \Gamma(E') \to \Gamma(E)$ is a C^∞-iso.

Otoh, let $\varphi: \Gamma(E) \xrightarrow{\cong} \Gamma(E')$ be a C^∞-iso.
Choose a complimentary bdle $E^\perp$ ($E \oplus E^\perp = M \times \mathbb{R}^n$)
It is easy to check that $\Gamma(E \oplus E^\perp) \cong$
$\cong \Gamma(E) \oplus \Gamma(E^\perp)$ & $\Gamma(E' \oplus E^\perp) \cong \Gamma(E') \oplus \Gamma(E^\perp)$,
so get a C^∞-iso

$$\Gamma(E) \oplus \Gamma(E^\perp) = \Gamma(E \oplus E^\perp) \xrightarrow{\cong} \Gamma(E' \oplus E^\perp) \cong \Gamma(E') \oplus \Gamma(E^\perp).$$

By the first step, $E \oplus E^\perp \cong E' \oplus E^\perp$ &
this isom. takes E into E', so

$$E \cong E'.$$

b) <u>Claim:</u> There is a canonical bijection

$$\Gamma(P \times_G X) \cong \{f: P \to X \mid f(pg) = \rho(g^{-1}) f(p)\}$$

where $\pi: P \to M$ is any G-princ. bdle &
X is a set on which G acts via
$$\rho: G \longrightarrow \mathrm{Aut}(X).$$

Pf: Let $s: M \to P \times_\rho X$, $m \mapsto [(p(m), x(m))]$
be a section, i.e. $\pi(p(m)) = m$.
Define $\varphi: P \to X$ by $\varphi(p) = x(\pi(p))$,
i.e. $\varphi(p)$ is the element of X s.t.
$$s(\pi(p)) := [(p, \varphi(p))].$$

1) φ is well-defined, i.e. $\varphi(p)$ is unique:
Assume $[(p, x_1)] = [(p, x_2)]$.
Then $\exists g \in G$ s.t. $pg = p$ & $\rho(g^{-1}) x_1 = x_2$.
From $pg = p$ & the freeness of the G-action get $g = e$ & so $x_1 = x_2$.

2) $\underline{\varphi \text{ is equivariant:}}$ $\quad s(m) = [(p, \varphi(p))] = [(pg, x)]$

iff $\varphi(p) = \rho(g^{-1})x$ by def. of $P \times_\rho X$

$$\Rightarrow \quad \varphi(pg) = x = \rho(g)\,\varphi(p).$$

$\rightsquigarrow$ Get map

$$\Gamma(P \times_\rho X) \longrightarrow \{f : P \to X \mid f(pg) = \rho(g) f(p)\},$$
$$s \longmapsto \varphi_s$$

Otoh, for given $\varphi : P \to X$ with $\varphi(pg) = \rho(g)\varphi(p)$, let

$s(\varphi) \in \Gamma(P \times_\rho X)$ be def. by

$s(\varphi)(m) := [(p, \varphi(p))]$ where $p \in \pi^{-1}(m)$

By equivariance, this is a well-def. section.

We have $\varphi = \varphi_{s(\varphi)}$ &

$$s = s(\varphi_s)$$

by construction, so the two maps are bijective & inverses of each other.

<u>Ex. 1 c)</u>: Let $p: E \to M$ be a vector bdle.

Choose open covering $\{U_i \mid i \in I\}$ of M on which E can be trivialized ($E|_{U_i} \cong U_i \times \mathbb{R}^n$)

Choose $\varphi_i : U_i \times \mathbb{R}^n \xleftarrow{\cong} E|_{U_i}$ & set

$$\tilde{\varphi}_{ij} : U_i \cap U_j \times \mathbb{R}^n \xrightarrow{\varphi_j^{-1}} E|_{U_i \cap U_j} \xrightarrow{\varphi_i} U_i \cap U_j \times \mathbb{R}^n.$$

Then $\tilde{\varphi}_{ij}$ is of the form

$$\tilde{\varphi}_{ij}(u, x) = (u, \varphi_{ij}(u) \cdot x)$$

where $\varphi_{ij} : U_i \cap U_j \longrightarrow Gl(\mathbb{R}^n)$

<u>Claim:</u> Such a tuple $(\varphi_{ij} : U_i \cap U_j \to Gl(\mathbb{R}^n))_{i,j \in I}$

is induced from a vector bdle iff the following hold:

$(*)$
- $\forall i \in I: \varphi_{ii} \equiv id_{\mathbb{R}^n}$
- $\forall i,j,k \in I: \varphi_{ik} = \varphi_{ij} \circ \varphi_{jk}$

Pf: (φ_{ij}) is induced from a v.b. Then

$$\varphi_{ii} = \varphi_i \circ \varphi_i^{-1} \equiv id_{\mathbb{R}^n}$$

& $\varphi_{ik} = (\varphi_i \circ \varphi_k^{-1}) = (\varphi_i \circ \varphi_j^{-1}) \circ (\varphi_j \circ \varphi_k^{-1}) =$

$$= \varphi_{ij} \circ \varphi_{jk}$$

Otoh, let (φ_{ij}) be a cocycle, i.e. satisfy $(*)$.

Set $E := \left(\coprod_{i \in I} (U_i \times \mathbb{R}^n) \right) \Big/ \left(\begin{array}{l} (u,v) \sim (u',v') \text{ iff} \\ u = u' \in U_i \cap U_j \ \& \\ \varphi_{ij}(v) = v' \text{ for some } i,j \end{array} \right)$

with the quotient topology induced from the disjoint union topology. Set $p : E \to M$,

$$[(u,v)] \mapsto u.$$

Then p is a v.b. whose ass. cocycle wrt $\{U_i \mid i \in I\}$ is (φ_{ij}):

A local trivialization of $E \to M$ is given by the canonical maps

$$U_i \times V \xrightarrow{\cong} p^{-1}(U_i) \subseteq \left(\coprod_{i \in I} (U_i \times V)\right)\Big/\sim,$$

$$(u,v) \longmapsto [(u,v)] \qquad \square$$

<u>Ex 1 d):</u> i) Let $\begin{array}{c} E \\ \downarrow \\ M \end{array}$, $\begin{array}{c} F \\ \downarrow \\ M \end{array}$ be vector bundles with fibers $E_m \cong V$, $F_m \cong W$ both trivialized over $\{U_i \mid i \in I\}$ &

Let (φ_{ij}) be the cocycle of E, (ψ_{ij}) the cocycle of F. Set

$$\varphi_{ij} \otimes \psi_{ij} : U_i \cap U_j \longrightarrow GL(V \otimes W),$$
$$u \longmapsto (v \otimes w \longmapsto (\varphi_{ij}(u) \cdot v) \otimes (\psi_{ij}(u) \cdot w))$$

Claim: $(\varphi_{ij} \otimes \psi_{ij})_{i,j}$ is a cocycle that is induced by a unique v.b., which we denote by $E \otimes F$.

Pf: That $(\varphi_{ij} \otimes \psi_{ij})$ is a cocycle is trivial.

Uniqueness: Assume G, H are vector bundles with the same associated cocycles $(\varphi_{ij} \otimes \psi_{ij})$. Then $G \cong H$ bc.:

$$\Gamma(G) \cong \{ (s_i : U_i \longrightarrow \tilde{V})_{i,j} \mid s_i|_{U_i \cap U_j}(u) = (\varphi_{ij} \otimes \psi_{ij})(u) \cdot s_j|_{U_i \cap U_j}(u) \}$$
$$\cong$$

$\Gamma(H)$,

so by (a), $H \cong G$ ☐

ii) Claim: If E, F are smooth bdles, then there is a canonical isom.

$$\Gamma(E \otimes F) \cong \Gamma(E) \otimes_{C^\infty} \Gamma(F)$$

Pf: If E & F are trivial, this is clear by (a).

If E, F are arbitrary, choose $E^\perp$ s.t. $E \oplus E^\perp$ is trivial & $F^\perp$ s.t. $F \oplus F^\perp$ is trivial.

Then $(E \oplus E^\perp) \otimes (F \oplus F^\perp) \cong (E \otimes F) \oplus$

$$\oplus (E \otimes F^\perp) \oplus (E^\perp \otimes F) \oplus (E^\perp \otimes F^\perp)$$

& we get

$$\Gamma(E\otimes F)\oplus\Gamma(E\otimes F^{\perp})\oplus\Gamma(E^{\perp}\otimes F)\oplus\Gamma(E^{\perp}\otimes F^{\perp})\cong$$

$$\cong\Gamma\big((E\oplus E^{\perp})\otimes(F\oplus F^{\perp})\big)\cong\Gamma(E\oplus E^{\perp})\otimes_{C^\infty}\Gamma(F\oplus F^{\perp})$$

$$\cong\big(\Gamma(E)\otimes_{C^\infty}\Gamma(F)\big)\oplus\big(\Gamma(E)\otimes_{C^\infty}\Gamma(F^{\perp})\big)\oplus\big(\Gamma(E^{\perp})\otimes_{C^\infty}\Gamma(F)\big)\oplus$$

$$\oplus\big(\Gamma(E^{\perp})\otimes_{C^\infty}\Gamma(F^{\perp})\big)$$ & one can check

that $\Gamma(E\otimes F)$ gets mapped into

$$\Gamma(E)\otimes_{C^\infty}\Gamma(F),$$

so $\Gamma(E\otimes F)\cong\Gamma(E)\otimes_{C^\infty}\Gamma(F)$. $\square$

(c.f. Milnor–Stasheff, Characteristic Classes, Ch. 3)

Ex. 1 e); $\pi: P \to M$ $\overset{\text{smooth}}{\curvearrowleft} G$ - principal bdle,

$$\rho: G \to Gl(V), \quad \rho': G \to Gl(W)$$

smooth representations of G.

Set.

$$\rho \otimes \rho' : G \to Gl(V \otimes W),$$

$$g \mapsto \big(v \otimes w \mapsto (\rho(g) \cdot v) \otimes (\rho'(g) \cdot w)\big)$$

(diagonal action). Then

Claim: $P \times_{\rho \otimes \rho'} (V \otimes W) = (P \times_\rho V) \otimes (P \times_{\rho'} W)$.

Pf: Choose a covering $\{U_i \mid i \in I\}$ of M s.t. $P|_{U_i} \cong U_i \times G$ via a section $g_i : U_i \to P|_{U_i}$. Then all three of

$P \times_\rho V$, $P \times_{\rho'} W$, $P \times_{\rho \otimes \rho'} (V \otimes W)$ are trivialized over

$\{U_i \mid i \in I\}$ & the ass. cocycles are

$$\varphi_{ij} : U_i \cap U_j \xrightarrow{g_i g_j^{-1}} G \xrightarrow{\rho} Gl(V)$$

$$\psi_{ij} : U_i \cap U_j \xrightarrow{g_i g_j^{-1}} G \xrightarrow{\rho'} Gl(W)$$

$$\eta_{ij} : U_i \cap U_j \xrightarrow{g_i g_j^{-1}} G \xrightarrow{\rho \otimes \rho'} Gl(V \otimes W).$$

It is obvious that $\eta_{ij} = \varphi_{ij} \otimes \psi_{ij}$,

so by definition $P \times_{\rho \otimes \rho'} (V \otimes W) = (P \times_{\rho} V) \otimes (P \times_{\rho'} W)$

Exercice 6

$$\begin{tikzcd} S^1 \arrow[r] & S^3 \arrow[d] \\ & S^2 \end{tikzcd}$$

$S^3 \subseteq \mathbb{C}^2$

$S^1 \subseteq \mathbb{C}$
unit cpx numbers

Here $S^1 \curvearrowright S^3$ is by isometries.

$\langle -,- \rangle_{\mathbb{C}^2}$ (sesquilinear) inner product gives a Riem. metric $\mathrm{Re}(\langle -,- \rangle_{\mathbb{C}})$ on $\mathbb{C}^2$, which we restrict to S^3.

$$\begin{aligned} &\mathrm{Re}\left(\left\langle \alpha \binom{z}{w}, \alpha \binom{z}{w} \right\rangle\right) \\ &= \mathrm{Re}\left(|\alpha|^2 \left\langle \binom{z}{w}, \binom{z}{w} \right\rangle\right) \end{aligned}$$

$\Longrightarrow$ get a connection by taking orthog. complement of VTP by Exercise 5.

$$T S^3_{(z,w)} = (z,w)^{\perp_{\mathbb{R}}}$$

for $(z,w) \in S^3$

<u>Pf</u>: γ a curve through (z,w)

$$0 = \frac{d}{dt}\Big|_{t=0} \underbrace{\langle \gamma(t), \gamma(t) \rangle_{\mathbb{C}}}_{\equiv 1}$$

$$= \langle \dot\gamma(0), \gamma(0) \rangle_{\mathbb{C}} + \langle \gamma(0), \dot\gamma(0) \rangle_{\mathbb{C}}$$

$$= 2\,\mathrm{Re}\,(\langle \dot\gamma(0), \underset{(z,w)}{\underset{\shortparallel}{\gamma(0)}} \rangle)$$

$\blacksquare$

$$VTS^3 = \langle i(z,w)\rangle_{\mathbb{R}} \quad \text{because}$$

$$\frac{d}{dt}\Big|_{t=0} (z,w)e^{it} = i\cdot(z,w)$$

$$\Rightarrow (VTS^3)^{\perp_{\mathbb{R}}} \text{ in } \mathbb{C}^2 \text{ is } i\cdot(z,w)^{\perp_{\mathbb{R}}}$$

$$\text{Above} \Rightarrow (VTS^3)^{\perp_{\mathbb{R}}} \cap TS^3$$

$$= \langle i(z,w), (z,w)\rangle_{\mathbb{R}}^{\perp_{\mathbb{R}}}$$

$$= \langle (z,w)\rangle_{\mathbb{C}}^{\perp_{\mathbb{C}}}$$

real span

$$\overset{(*)}{=} \langle (\bar{w},-\bar{z}), i(\bar{w},-\bar{z})\rangle_{\mathbb{R}}$$

$$= \langle (\bar{w},-\bar{z})\rangle_{\mathbb{C}}$$

(*) because

$$\langle (\bar{w}, -\bar{z}), (z, w) \rangle$$

$$= wz - zw = 0$$

So $H_A = (VTS^3)^{\perp_{\mathbb{R}}} \cap TS^3$

$$= \left\langle \begin{pmatrix} \bar{w} \\ -\bar{z} \end{pmatrix}, i \begin{pmatrix} \bar{w} \\ -\bar{z} \end{pmatrix} \right\rangle_{\mathbb{R}}$$

What is the connection 1-form ω_A?

Needs to satisfy

$$\begin{cases} \omega_A(i(z,w)) \overset{!}{=} i \\ \text{and} \\ \omega_A \big|_{H_A} \overset{!}{\equiv} 0 \end{cases}$$

$$\omega_A = i\,\mathrm{Re}\left(\left\langle i\binom{z}{w}, - \right\rangle_{\mathbb{C}}\right)$$

<u>works:</u>

- $$\omega_A\left(\binom{\bar w}{-\bar z}\right) = i\,\mathrm{Re}\left(\left\langle i\binom{z}{w}, \binom{\bar w}{-\bar z}\right\rangle_{\mathbb{C}}\right)$$
  $$= i\,\mathrm{Re}\left(-i\left(\bar z\,\bar w - \bar w\,\bar z\right)\right)$$
  $$= 0$$

- Similar:
  $$\omega_A\left(i\binom{\bar w}{-\bar z}\right) = 0$$

- $$\omega_A\left(i\binom{z}{w}\right)$$
  $$= i\,\mathrm{Re}\left(\left\langle i\binom{z}{w}, i\binom{z}{w}\right\rangle\right)$$
  $$= i\,\mathrm{Re}\Big(\underbrace{(|z|^2+|w|^2)}_{=1}\Big)$$
  $$= i.$$

In fact

$$\omega_A = i \cdot \operatorname{Im}\left(\left\langle \binom{z}{w}, - \right\rangle\right)$$

is a diff. expression for the same $i\mathbb{R}$-valued 1-form.

In diff-form notation:

$$\omega_A = i \operatorname{Im}\left(\bar{z}\,dz + \bar{w}\,dw\right)$$

(Rk:

$$\omega_A = \bar{z}\,dz + \bar{w}\,dw$$

if restricted to TS^3, but it's not an $i\mathbb{R}$-valued 1 form on $T\mathbb{C}^2$)

Exercise:

On $S^{2n+1} \subseteq \mathbb{C}^{2n+1}$

$$\begin{array}{ccc} S^1 \longrightarrow S^{2n+1} =: P & & \text{coo. } (z_0, \ldots, z_n) \\ \downarrow & & \\ \mathbb{C}P^n & & \end{array}$$

$$VTP_{(z_0,\ldots,z_n)} = \langle i\,(z_0, \ldots, z_n) \rangle_{\mathbb{R}}$$

and

$$\omega_A = \sum_{i=0}^{n} \overline{z_i}\, dz_i$$

$$= i \operatorname{Im} \left(\sum_{i=0}^{n} \overline{z_i}\, dz_i \right)$$

and

$$H_A = \langle (z_0, \ldots, z_n) \rangle_{\mathbb{C}}^{\perp_{\mathbb{C}}}$$