$$\begin{array}{ccc} G & \to & P \\ & & \downarrow \\ & & M \end{array}$$

A a connection on P

Reminder:

Parallel transport in P along path $\gamma : [0,1] \to M$ is a lift

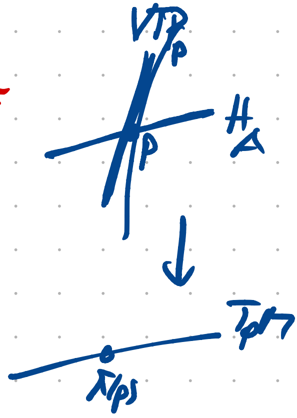
$$\tilde{\gamma}_p : [0,1] \to P$$

s.th. $\pi \circ \tilde{\gamma}_p = \gamma$

$$\dot{\tilde{\gamma}}_p(t) \in H_A \quad \forall t$$

and

$$\tilde{\gamma}_p(0) = p$$



$$d\pi : H_A \xrightarrow{\cong} T_pM$$

$$\boxed{\begin{array}{l} Par_\gamma^A : P_{\gamma(0)} \to P_{\gamma(1)} \\ \qquad u \mapsto \tilde{\gamma}_u(1) \\ \text{is } G\text{-invariant} \end{array}}$$

Par. transport in

$$E = P \times_G V :$$

$$Par_\gamma^{A,E} : E_{\gamma(0)} \to E_{\gamma(1)}$$

$$[p, v] \mapsto [Par_\gamma^A(p), v]$$

well-defined because of G-invariance

$$Par_\gamma^A(pg) = Par_\gamma^A(p)\, g$$

$$\begin{tikzcd}
\Omega^*(P;V)_{\text{hor. eq.}} \arrow[r, "d_A"] \arrow[d, "\cong"'] & \Omega^*(P;V)_{\text{hor. eq.}} \arrow[d, "\cong"] \\
\Omega^*(M; P\times_G V) \arrow[r, "\bar{d}_A"] & \Omega^*(M; P\times_G V)
\end{tikzcd}$$

(Def^n of $\bar{d}_A$) Saw: $\bar{d}_A$ satisfies Leibniz rule

$$d_A \overset{\text{Def}^n}{=} d \circ pr_{H_A}$$

$$\nabla^A := \bar{d}_A \big|_{\Gamma(P\times_G V)}$$ is a cov. derivative (class).

Parallel transport of a cov. derivative:

$E \longrightarrow M \ni \gamma$ ∇ cov. deriv.

$s \in \Gamma(E)$

If

$$(\nabla_{\dot\gamma(t)} s)(\gamma(t)) = 0 \qquad (*)$$

then $s(\gamma(t)) \in E_{\gamma(t)}$ is said to arise from parallel transport along γ from $s(\gamma(0)) \in E_{\gamma(0)}$.

$\rightsquigarrow \; Par_\gamma^\nabla \in \mathrm{Isom}(E_{\gamma(0)}, E_{\gamma(1)})$

Prop: $Par_\gamma^{A,E} = Par_\gamma^{\nabla^A}$

$E = P \times_\rho V$

$\rho: G \to \mathrm{Aut}(V)$ group hom.

$$\text{Notation}\left[\begin{array}{c} \Gamma(M; P\times_\rho V) \\ \hat{\ } \downarrow (\cong) \uparrow \check{\ } \\ \Gamma_\rho(P; V) \end{array}\right]$$

$\tilde{\gamma}$ horiz. lift

Proof:

$$\left(\nabla^A_{\dot\gamma(t)} s\right)(\gamma(t))$$

$$\overset{\text{Def}^{\text{n}}}{=} \left(\overline{d_A s}\right)(\dot\gamma(t))$$

$$= \overline{d_A \hat{s}\,(\dot{\tilde\gamma}(t))}$$

here really we take an arbitrary lift projecting onto $\dot\gamma(t)$

$$= [\tilde\gamma(t), d_A \hat{s}\,(\dot{\tilde\gamma}(t))]$$

$$= [\tilde\gamma(t), ds(\dot{\tilde\gamma}(t))]$$

$$\overset{\text{Def}^{\text{n}}}{=} [\tilde\gamma(t), \tfrac{d}{dt} s(\tilde\gamma(t))]$$

$$\left(\nabla^A_{\dot\gamma(t)} s\right)(\gamma(t))$$

$$\overset{\text{Def}^n}{=} \left(\overline{d_A s}\right)(\dot\gamma(t))$$

$$= \overline{d_A \hat{s}\,(\dot{\tilde\gamma}(t))}$$

here really we take an arbitrary lift projecting onto $\dot\gamma(t)$

$$= \left[\tilde\gamma(t),\, d_A \hat{s}\,(\dot{\tilde\gamma}(t))\right]$$

$$= \left[\tilde\gamma(t),\, d\hat{s}\,(\dot{\tilde\gamma}(t))\right]$$

$$\overset{\text{Def}^n}{=} \left[\tilde\gamma(t),\, \frac{d}{dt}\hat{s}\,(\tilde\gamma(t))\right]$$

$$\overset{\text{Def}^n}{=} \left[\mathrm{Par}^A_\gamma(\tilde\gamma(0)),\, \frac{d}{dt}\underbrace{\hat{s}\,(\tilde\gamma(t))}_{\text{path in}}\right] \checkmark$$

$$\left(\overset{\text{Def}^n}{=} \underbrace{\mathrm{Par}^{A,E}_{\gamma_t}}_{\text{path from } \gamma(0) \text{ to } \gamma(t)}\left(\left[\tilde\gamma(0),\, \frac{d}{dt}\hat{s}\,(\tilde\gamma(t))\right]\right)\right)$$

$$\boxed{\left(\nabla^A_{\dot\gamma(t)} s\right)(\gamma(t)) = 0 \iff \frac{d}{dt}\hat{s}\,(\tilde\gamma(t)) = 0 \quad \forall t\in[0,1]}$$

$$\left.\begin{array}{l} \text{If } \frac{d}{dt}\hat{s}(\tilde{\gamma}(t)) = 0 \quad \forall t, \text{ then} \\ \quad \hat{s}(\tilde{\gamma}(1)) = \hat{s}(\tilde{\gamma}(0)) \end{array}\right] (*)$$

$$s(\gamma(1)) = \mathrm{Par}_\gamma^{\nabla^A}(s(\gamma(0)))$$

$$\text{if } \nabla^A_{\dot\gamma(t)} s \equiv 0$$

Often

$$\underbrace{\mathrm{Par}_\gamma^{A,E}(s(\gamma(0)))}_{[\tilde{\gamma}(0), \hat{s}(\tilde{\gamma}(0))]} = [\mathrm{Par}_\gamma^{A}(\tilde{\gamma}(0)), \hat{s}(\tilde{\gamma}(0))]$$

$$= [\tilde{\gamma}(1), \hat{s}(\tilde{\gamma}(0))]$$

$$\overset{\text{if } (*)}{\underset{\text{holds}}{=}} [\tilde{\gamma}(1), \hat{s}(\tilde{\gamma}(1))]$$

$$= s(\gamma(1))$$

$\blacksquare$

$$\Rightarrow \mathrm{Par}_\gamma^{A,E}(s(\gamma(0))) = s(\gamma(1)) \text{ if}$$

$$(*) \text{ holds} \iff (\nabla^A_{\dot\gamma(t)} s)(\gamma(t)) \equiv 0$$

Often $\mathrm{Par}_\gamma^{\nabla^A}(s(\gamma(0))) = s(\gamma(1))$ also, by Def. of $\mathrm{Par}_\gamma^{\nabla^A}$

<u>Recall</u> $E \to M$ a vector bundle

$$R^\nabla(v,w) = \nabla_v \nabla_w - \nabla_w \nabla_v - \nabla_{[v,w]}$$

defines <u>curvature</u>

$$R^\nabla \in \Gamma(\Lambda^2 T^*M \otimes \mathrm{End}(E))$$

<u>Prop:</u> If $E = P \times_G V$, A a connection on P,

then

$\rho_* : \mathfrak{g} \to \mathrm{End}(V)$

$$\begin{tikzcd}
\Gamma_G^\tau(P;V) \arrow[r, "\rho_* \Omega_A"] \arrow[d, "-"', "\cong"] & \Gamma_G^\tau(P;V) \arrow[d, "-"', "\cong"] \\
\Gamma(E) \arrow[r, "R^{\nabla^A}"] & \Gamma(E)
\end{tikzcd}$$

<u>Proof:</u> If $\varphi \in \Gamma(P \times_G V)$, $\hat{\varphi}$ lift to $\in \Gamma_G^\tau(P;V)$

$(\ast)$ $$(\nabla^A_v \varphi)(x) = [p, (d_A \hat{\varphi})_p(\tilde{v})]$$ where $d\pi(\tilde{v}) = v$

$\Big(\varphi(x) = [p, \hat{\varphi}(p)]$

where $\pi(p) = x \Big)$

$$\Rightarrow \left(R^{\nabla^A}_x(v_x, w_x)\,\varphi\right)(x)$$

$$= R^{\nabla^A}_x(v_x, w_x)\,[p, \hat\varphi(p)]$$

v, w vector fields, $\tilde v, \tilde w$ horizontal lifts wrt $d\pi|_{H_A} : H_A \to TM$

$$\overset{(*)}{=} \Big[p, \big(\tilde v_p.(\tilde w.\hat\varphi) - \tilde w_p.(\tilde v.\hat\varphi) - \widetilde{[v,w]}_p.\hat\varphi\big)(p)\Big]$$

$$= \Big[p, \underbrace{\big(\underbrace{[\tilde v, \tilde w]}_{\text{does not need to be horiz (not in general if } H_A \text{ is not involutive)}}.\hat\varphi - \widetilde{[v,w]}_p.\hat\varphi\big)}_{\text{dif } \in \ker(d\pi|) = VTP}(p)\Big]$$

$$\boxed{\begin{aligned} &\text{but} \\ &d\pi([\tilde v, \tilde w]) \\ &= [d\pi(\tilde v), d\pi(\tilde w)] \\ &= [v, w] \\ &= d\pi(\widetilde{[v,w]}) \end{aligned}}$$

$$= \Big[p, \big(\pi_V([\tilde v, \tilde w]).\hat\varphi\big)(p)\Big]$$

$$\overset{\text{by defn of } \omega_A}{=} \Big[p, \omega_A([\tilde v, \tilde w])^{\#}_p.\hat\varphi\Big]$$

$$= -\Big[p, \Omega_A(\tilde v, \tilde w)^{\#}.\hat\varphi\Big]$$

$d_A\omega_A = \Omega_A$

$\Rightarrow d_A\omega_A(\tilde v, \tilde w)$

$\overset{\shortparallel}{d\omega_A}(\tilde v, \tilde w)$

$-\omega_A([\tilde v, \tilde w])$

$+ \underbrace{\tilde v\,\omega_A(\tilde w) - \tilde w\,\omega_A(\tilde v)}_{=0}$

$$\Omega_A(\tilde v, \tilde w)^{\#}\hat\varphi(p)$$

$$= \frac{d}{dt}\Big|_{t=0} \hat\varphi\left(p\, e^{t\Omega_A(\tilde v, \tilde w)}\right)$$

$$\overset{\text{Equiv. of } \bar{\bar\varphi}}{=} \frac{d}{dt}\Big|_{t=0} S\left(e^{-t\Omega_A(\tilde v, \tilde w)}\right)\hat\varphi(p)$$

$$= -S_*\left(\Omega_A(\tilde v, \tilde w)\right)\hat\varphi(p)$$

$$= [p, S_*(\Omega_A(\tilde v, \tilde w))\hat\varphi(p)]$$

$$\Longrightarrow \boxed{\begin{aligned} R_x^{\nabla^\Delta}(v_x, w_x)[p, \hat\varphi(p)] \\ = [p, S_*(\Omega_A(\tilde v, \tilde w))\hat\varphi(p)] \end{aligned}}$$

$\blacksquare$