Ex. 2: Compute the par. transport for the connection on $[0,1] \times i\mathbb{R} \to [0,1]$ described by the form $d + a$, where $a \in \Omega^1([0,1], i\mathbb{R})$

Pf: $[0,1] \times i\mathbb{R}$ is the trivial bdle, there is a $C^\infty(I)$-linear iso

$$\Omega^1([0,1], i\mathbb{R}) \longrightarrow C^\infty([0,1], i\mathbb{R})$$

given by $a \longmapsto a(\partial_t)$

(where $t : [0,1] \hookrightarrow \mathbb{R}$ is the inclusion).

A section of $[0,1] \times \mathbb{C} \to [0,1]$ is then covariant constant iff

$$\frac{d}{dt} s + a(\partial_t) s = 0$$

This is a linear ODE with unique solution given by

$$s(t) = \exp\left(-\int_0^t a(\partial_t)\,dt\right) \cdot s(0)$$

So, $s(1) = \exp\left(-\int_0^1 a(\partial_t)\,dt\right) \cdot s(0)$

& thus

$$\mathrm{Hol}(d+a) = \exp\Big(\underbrace{-\int_0^1 a(\partial_t)\,dt}_{\in i\mathbb{R}}\Big) \in S^1 \qquad \blacksquare$$

<u>Ex 3</u>

a) $S^3/C_n \to S^2$ is a principal S^1-bundle

Pf: <u>Generally:</u> if $H \trianglelefteq G$ is a normal Lie subgroup & $\pi: P \to M$ is a principal G-bundle, then

$$\pi_H : P/H \longrightarrow M$$

is a principal (G/H)-bundle:

<u>Pf of this:</u> It is clear that G/H acts freely & smoothly on P/H & that this action is transitive on fibres.

If $U \subseteq M$ is open & $\phi : \pi^{-1}(U) \to U \times G$ is a trivialization of π over U, then by G-equivariance, ϕ induces a trivialization of π_H via the comm. diagram

$$\begin{tikzcd}
\pi^{-1}(U)/H \arrow[r, "\cong", "\phi/H"'] \arrow[d] & (U \times G)/H \arrow[d] \\
\pi_H^{-1}(U) \arrow[r, "\cong", "\phi_H"'] & U \times (G/H).
\end{tikzcd}$$

Thus, the claim.

In the case at hand we have $G = S^1$ & $H = C_n$ & the claim follows by observing that $S^1/C_n \cong_{\text{diff}} S^1$ via

$$[z] \longmapsto z^n \qquad \blacksquare$$

b) By def. $c_1(S^3/C_n \times_{S} \mathbb{C}) := c_1(S^3/C_n \to S^2)$.

We use Chern-Weil-theory to compute the characteristic classes/numbers:

We have already seen that the restr. of $\bar{z}_1 dz_1 + \bar{z}_2 dz_2$ defines a connection A on the Hopf bdle $S^3 \longrightarrow S^2$ with $c_1(S^3 \to S^2) = -1$.

By lecture, A induces a connection A_n on $S^3/C_n \to S^2$ whose curvature satisfies

$$f^* \Omega_{A_n} = \lambda_{n,*} \circ \Omega_A = n \cdot \Omega_A$$

where $f: S^3 \to S^3/C_n$ &

$$\lambda_{n,*} : i\mathbb{R} \to i\mathbb{R},\ it \mapsto n \cdot it$$

$\left(\lambda_{n,*} = \text{derivative of } S^1 \to S^1, z \mapsto z^n \right)$

Now let $X, Y \in \Gamma(TS^2)$ & let $\tilde{X}, \tilde{Y} \in \Gamma(TS^3)$ be arbitrary lifts of X, Y ($d\pi(\tilde{X}) = X$, $d\pi(\tilde{Y}) = Y$). Then

$$f_* \tilde{X}, f_* \tilde{Y} \in \Gamma(T S^3/C_n)$$

are lifts of X, Y along $S^3/C_n \to S^2$.

We get

$$F_{A_n}(X,Y) = \Omega_{A_n}(f_*\tilde{X}, f_*\tilde{Y}) =$$

$$= f^*\Omega_{A_n}(\tilde{X},\tilde{Y}) =$$

$$= n \cdot \Omega_A(\tilde{X},\tilde{Y}) =$$

$$= n \cdot F_A(X,Y).$$

$$\Rightarrow \quad c_1(S^3/C_n \to S^2) = \left[-\frac{i}{2\pi}F_{A_n}\right] =$$

$$= n \cdot \left[-\frac{i}{2\pi}F_A\right] =$$

$$= n \cdot c_1(S^3 \to S^2).$$

In particular, the chern number of S^3/C_n is

$$\langle c_1(S^3/C_n \to S^2), [S^2] \rangle = -n$$

bc. the char. number of the Hopf bdlle is -1.

c) The homomorphism of S^1-principal bdles

$$\begin{pmatrix} S^3 \\ \downarrow \\ S^2 \end{pmatrix} \longrightarrow \begin{pmatrix} S^3/C_n \\ \downarrow \\ S^2 \end{pmatrix}$$

is of type $\lambda_n : S^1 \longrightarrow S^1$, $z \longmapsto z^n$

As we have seen in a previous exercise,

$$H^{\otimes k} = S^3 \times_{g_k} \mathbb{C} \quad \text{where } g_k : S^1 \longrightarrow \mathrm{Aut}(\mathbb{C}), \quad z \longmapsto \mathrm{mult}_{z^k}.$$

We can write this as

$$\varrho_k = S^1 \xrightarrow{\lambda_k} S^1 \xrightarrow{\varrho} \mathrm{Aut}(\mathbb{C}) \text{ \& we get}$$

$$H^{\otimes k} \cong S^3 \underset{\varrho_k}{\times} \mathbb{C} \cong S^3 \underset{\varrho \circ \lambda_k}{\times} \mathbb{C} \underset{\substack{\uparrow \\ \text{lecture}}}{\cong} (S^3/C_k) \underset{\varrho}{\times} \mathbb{C}$$

∎

<u>Ex 4:</u> Let $f: S^2 = \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^n$. Then

f induces isomorphisms

$$f^*: H^2(\mathbb{C}P^n) \xrightarrow{\cong} H^2(S^2)$$

$$f_*: H_2(S^2) \xrightarrow{\cong} H_2(\mathbb{C}P^n)$$

& we get

$$c_1\left(S^{2n+1}/C_k \xrightarrow{h_{n,k}} \mathbb{C}P^n\right) =$$

$$= (f^*)^{-1}\left(f^* c_1(h_{n,k})\right) \overset{\substack{\text{Naturality of } c_1 \\ \downarrow}}{=}$$

$$= (f^*)^{-1}\left(c_1(f^* h_{n,k})\right).$$

To compute $c_1(f^* h_{n,k})$, we note that $f^* h_{n,1}$ is the Hopf bundle & the restriction of $C_k \curvearrowright S^{2n+1}$ to $S^3 \subseteq S^{2n+1}$ is the same action as in exercise 3.

So we get $f^* h_{n,k} = (S^3/C_k \longrightarrow S^2)$

& thus, b.c. $(f^*)^{-1}(1) = 1$, we get that $h_{n,k}$ $S^{2n+1}/C_k \longrightarrow \mathbb{C}P^n$ has chern class

$$c_1(h_{n,k}) = (f^*)^{-1} c_1(S^3/C_k \longrightarrow S^2)$$

& chern number $-k$. $\blacksquare$

<u>Ex 5</u>: $\pi : P \to M$ is a G-principal bundle, P carries a Riem. metric

& $G \curvearrowright P$ via isometries, then

The pointwise orthogonal complement of VTP defines a connection. H

Pf: It is clear that for every $p \in P$,

$$H_p \oplus VTP_p = (VTP_p)^{\perp} \oplus VTP_p = T_pP.$$

Because taking orthogonal complements is a smooth operation (Gram-Schmidt), $H \subseteq TP$ is a subbundle.

Need to check that for every $g \in G$,

$$dR_g(H) = H.$$

Because $G \curvearrowright P$ by isometries,

$$dR_g : T_p P \longrightarrow T_{pg} P$$

is an isometry of Hilbert spaces. Hence,

$$dR_g\left(VTP_p^{\perp}\right) = \left(VTP_{pg}\right)^{\perp} = H_{pg} \qquad \blacksquare$$

c_1 is determined by the map

$$\begin{array}{c} c_1 : H_2(\mathbb{CP}^n) \longrightarrow \mathbb{R} \\ \quad \Vert \\ \quad \langle [\mathbb{CP}^1] \rangle \end{array}$$

$$\overbrace{\ast F_A}^{\text{self dual}} = F_A$$

$$\rightsquigarrow \quad c_2(E \to M) = \|F_A\|_{L^2}$$

Anti-self dual:

$$*F_A = -F_A$$

$$\rightsquigarrow c_2(E \to M) = -\|F_A\|_{L^2}$$

$$\|F_A\|_{L^2} = -\int_M \mathrm{tr}(F_A \wedge F_A) =$$

$$= \int_M \mathrm{tr}(F_A \wedge * F_A) =$$

if A is ASD

$$= c_2(A)$$

<u>Upshot</u>: If $c_2(E) \neq 0$, then E cannot have both self-dual & anti-self-dual connections.

Thus: If both P_+ & P_- are top. nontriv. ($\Leftrightarrow c_2(P_+), c_2(P_-) \neq 0$), then we get $P_+ \not\cong P_-$ if we find an ASD conn. on one & an SD conn. on the other.

<u>Ex 7</u>: a) <u>Claim</u>: $\mathbb{HP}^1 \cong_{diff} S^4$.

<u>Pf</u>: S^4 is diffeomorphic, via stereographic projection, to $\mathbb{H} \cup \{\infty\}$ with the diff str. defined by the charts

$$q = id_{\mathbb{H}} : \mathbb{H} \longrightarrow \mathbb{H} \cong \mathbb{R}^4$$

& $$q^{-1}: (\mathbb{H} \setminus \{0\}) \cup \{\infty\} \longrightarrow \mathbb{H},$$

$$q \longmapsto \begin{cases} q^{-1}, & \text{if } q \neq \infty \\ 0, & \text{if } q = \infty \end{cases}$$

The map

$$\mathbb{H}\times\mathbb{H} \supseteq S^7 \xrightarrow{\varphi} \mathbb{H}\cup\{\infty\},$$

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \longmapsto \begin{cases} q_1 q_2^{-1}, & \text{if } q_2 \neq 0 \\ \infty, & \text{if } q_2 = 0 \end{cases}$$

is S^3-invariant $\left(\varphi\begin{pmatrix} q_1\cdot u \\ q_2\cdot u\end{pmatrix} = (q_1 u)(q_2 u)^{-1} = \right.$

$$\left. = q_1 \cancel{(u\cdot u^{-1})} q_2^{-1} = \varphi\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}\right),$$

surjective (bc. it is continuous, $\mathbb{H}\cup\{\infty\}$ is connected & S^7 is compact)

& is smooth & thus induces a smooth surj.

$$S^7/S^3 \longrightarrow \mathbb{H}\cup\{\infty\}.$$

A smooth inverse is given by

$$q \longmapsto \begin{cases} [q:1], & q \neq \infty \\ [1:0], & q = \infty \end{cases}$$

$\square$