Prop: Let X be a smooth closed, conn., oriented 4-mfd. Then $SU(2)$-bundles over X are completely determined by their second Chern class.

Recall: If $P \to X$ is a principal $SU(2)$-bundle, & A is a connection on P with curvature form $F_A \in \Omega^2(X; \operatorname{ad} P)$ then the class $(\ast) \ \frac{1}{2\pi}\left[\operatorname{tr}(F_A \wedge F_A)\right] \in \Omega^2_{dR}(X)$ is called (real) second Chern class

More general: $U(n)$-bundles over a finite dim. mfd X are determined by htpy classes of maps

$$[X, BU(n)]$$

$$H^*(BU(n);\mathbb{Z}) = \mathbb{Z}[c_1,\dots,c_n],$$
$$c_i \in H^{2i}(BU(n))$$

Ex: $BU(1) = \mathbb{C}P^\infty$ &

$$H^*(\mathbb{C}P^\infty) = \mathbb{Z}[x]$$

If $E \to X$ is the pullback of the ~~the~~ tautological bundle

$$EU(n) \to BU(n)$$

via $f_E : X \to BU(n)$

Then
$$c_i(E) := f_E^* c_i$$
$\lrcorner$

$BSU(2) = \mathbb{HP}^\infty$, $\mathbb{HP}^\infty$ has a CW-structure $\mathbb{HP}^0 = \{pt\} \subseteq \mathbb{HP}^1 = S^4 \subseteq \mathbb{HP}^2 \subseteq \dots$

with exactly one cell in every dim. $4n$.

Now, X has a CW-structure with exactly one 4-cell & no higher dim. cells.

By cellular approx.,
$$[X, \mathbb{HP}^\infty] = [X, \mathbb{HP}^1] =$$
$$= [X, S^4]$$

Denote the 3-skeleton of X by $X^3 \subseteq X$

$\rightsquigarrow$ $X/X^3 = S^4$.

Using the CW-decomp. $\{pt\} \subseteq S^4$

& cell. approx. get

$$[X, S^4] = [X/X^3, S^4] = [S^4, S^4]$$

The latter is $\cong \mathbb{Z}$ via degree.

The tautological bdle over $\mathbb{HP}^1$ is isomorphic to the $\mathbb{H}$-dual Λ^* of the universal subbundle $\Lambda \subseteq \mathbb{HP}^1 \times \mathbb{H}^2$ def. by $\Lambda_p = \left\{ \left(\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, p \right) \,\middle|\, \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \in p \right\}$

Let $p = [X:Y] \in \mathbb{H}P^1$. Then projection onto the first coordinate

$$s(p) : \Lambda_p \longrightarrow \mathbb{H}$$

is a linear map that depends smoothly on p, so it defines a section s of

$$\Lambda^* = \mathrm{Hom}_{\mathbb{H}}(\Lambda, \underline{\mathbb{H}})$$

<u>Fact:</u> $$c_2(\Lambda^*)[\mathbb{H}P^1] = \sum_{x \in s^{-1}(\{0\})} (-1)^{\varepsilon}$$

where $s : \mathbb{H}P^1 \longrightarrow \Lambda^*$ is a transverse (to the zero section) section

s is transverse & has exactly one zero, namely $[0:1] \in \mathbb{H}P^1$.

Thus, $c_2(\mathbb{1}^*)[\mathbb{H}P^1] = \pm 1$

& then we get $c_2(\mathbb{1}^*) = \pm PD([\mathbb{H}P^1])$

Now, let $E \to X$ be an $SU(2)$-bdle with classifying map

$$f_E : X \to S^4.$$

Then

$$c_2(E) = f^* c_2(\mathbb{1}^*) = \pm f^* PD([\mathbb{H}P^1]) =$$

$$= \deg f_E \cdot PD([X])$$

$\Rightarrow$ $c_2(E)$ determines the bundle.

<u>More generally:</u> $U(n)$-bundles over 4-mfds

are determined by c_1, c_2

(Steenrod, "Topology of fiber bundles")

A 4-classifying bdle for $U(2)$

$$\begin{array}{ccc} V_{4,2}(\mathbb{C}) & \longrightarrow & Gr_{4,2}(\mathbb{C}) \\ \shortparallel & & \shortparallel \\ U(4)/U(2) & & U(4)/U(2)\times U(2) \end{array}$$

<u>Warning:</u> This is only true in dim ≤ 4,

e.g.: There is a homotopically nontriv. map

$$S^5 \longrightarrow S^4$$

giving a nontrivial $SU(2)$-bundle $E \to S^5$.

But $c_2(E) \in H^4(S^5) = \{0\}$ Tib

Prop.: Let X be a closed, smooth, conn., oriented 4-mfd. & $E \to X$ a $U(2)$-bundle. Let $su(E)$ be the bundle of trace-free, skew-symm. endomorphisms of E. Then

$$p_1(su(E)) = c_1(E)^2 - 4c_2(E).$$

Pf.: By def.

$$p_1(su(E)) := -c_2\left(su(E) \otimes_{\mathbb{R}} \mathbb{C}\right) =$$

$$= -c_2\Big(\underbrace{\mathrm{End}_0(E)}_{=\text{ trace-free endomorphisms}}\Big) =$$

$$= -c_2\left(\mathrm{End}_0(E) \oplus \underline{\mathbb{C}}\right) =$$

$$c(E) := \sum_{i=0}^{\infty} c_i(E)$$ "total chern class"

$$\rightsquigarrow \quad c(E \oplus F) = c(E) \cup c(F)$$

$$c(\mathbb{C}^n) = 1$$

$$= -c_2(\mathrm{End}(E)) =$$

$$= -c_2(E \otimes E^*)$$

Introduce the following:

Formally write $c_i(E) = \sigma_i(x_1, x_2)$

where σ_i is the ith elementary symm. polynomial

If $E \to X$ is a cx v.bundle,
There is a space Q_E with
a map $\pi_E : Q_E \to E$ s.t.

1) $H^*(E) \to H^*(Q_E)$
is injective

2) $\pi_E^* E$ splits as a direct
sum of line bundles,

$$\pi_E^* E = \bigoplus_{i=1}^{n} L_i$$

"splitting principle"

($BU(1)^n \longrightarrow BU(n)$ has this property
wrt. the universal bundle)

Then $x_i = c_1(L_i)$ &

$$\pi_E^* c_i(E) = \sigma_i(x_i)$$

holds exactly.

& set

$$ch(E) := \exp x_1 + \exp x_2 =$$

$$= \sum_{n=0}^{\infty} \frac{x_1^n}{n!} + \sum_{n=0}^{\infty} \frac{x_2^n}{n!} \in$$

$$H^*(X; \mathbb{Q})$$

This is called Chern character

Explicitly, using $c_1(E) = x_1 + x_2$,
$c_2(E) = x_1 x_2$,

we compute

$$ch(E) = 2 + c_1(E) + \frac{1}{2}\left(c_1(E)^2 - 2c_2(E)\right).$$

This is useful bc.

$$ch(E \otimes E^*) = ch(E) \cdot ch(E^*)$$

$$\left(\& \; ch(E \oplus F) = ch(E) + ch(F)\right)$$

In our case, using $c_i(E^*) = (-1)^i c_i(E)$,

we compute:

$$4 + c_1(E\otimes E^*) + \frac{1}{2}\left(c_1(E\otimes E^*)^2 - 2c_2(E\otimes E^*)\right) =$$

$$= ch(E\otimes E^*) = ch(E)\cdot ch(E^*) =$$

$$= \left(2 + c_1(E) + \frac{1}{2}\left(c_1(E)^2 - 2c_2(E)\right)\right)\cdot$$

$$\cdot\left(2 + c_1(E^*) + \frac{1}{2}\left(c_1(E)^2 - 2c_2(E)\right)\right) =$$

$H^i(X) = 0$ for $i \geq 4$

$$= 4 + \left(c_1(E)^2 - 4c_2(E)\right).$$

Now, comparing degreewise, we first get

$$c_1(E\otimes E^*) = 0 \quad \&$$

then

$$-c_2(E\otimes E^*) = \frac{1}{2}\left(0 - 2c_2(E\otimes E^*)\right) =$$

$$= \frac{1}{2}\left(c_1\left(\underline{E} \otimes \underline{E}^*\right)^2 - 2c_2\left(\underline{E} \otimes \underline{E}^*\right)\right) =$$

$$c_1\left(\underline{E}\right)^2 - 4c_2\left(\underline{E}\right) \qquad \blacksquare$$