Sobolev spaces / norms $W^{k,p}(U)$

<u>Problem:</u> On L^2, cannot differentiate

<u>Solution:</u> Sobolev norms & completions

<u>Def:</u> $u, v \in L^1_{loc}(U)$ $(U \subseteq \mathbb{R}^n$ open$)$

$$L^1_{loc}(U) = \{[f] \mid \forall x \in U \ \exists V \ni x \text{ s.t. } \|f\|_{L^1(V)} < \infty\}$$

$$\|f\|_{L^1(V)} = \int_V |f| \, d\mu$$

Then for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ we write $v = D^\alpha u$ if for every smooth, compactly supp. $\phi : U \to \mathbb{R}$,

$\left[|\alpha| := \sum \alpha_i \right]$

$$\int_U u D^\alpha \phi \, d\mu = (-1)^{|\alpha|} \int_U v \phi \, d\mu$$

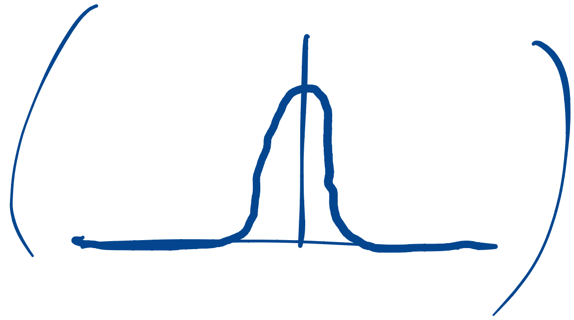
where $\displaystyle D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$

Define $W^{k,p}(U)$ as the set of L^1_{loc}-fcts u s.t. for all α w/ $|\alpha| \leq k$ $D^\alpha u$ exists & lies in L^p.

<u>Fact 1:</u> $W^{k,p}(U)$ is the completion of $C^\infty(U) \cap W^{k,p}(U)$ wrt the norm

$$\|\varphi\|_{p,k} := \sum_{|\alpha| \leq k} \|D^\alpha \varphi\|_{L^p(U)}$$

Γ ε bump function

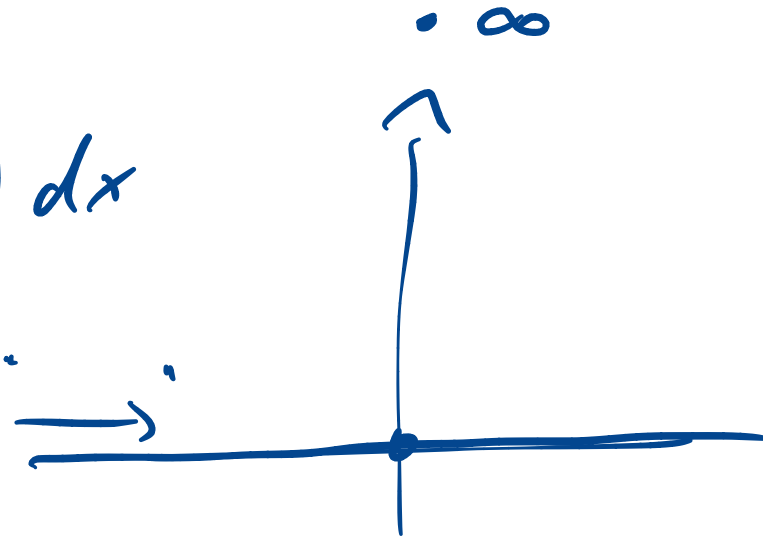


$\rightsquigarrow \varepsilon * u \in C^\infty$

$$\parallel$$

$$(\varepsilon * u)(y) = \int_{\mathbb{R}^n} \varepsilon(y)\, u(x-y)\, dx$$

δ for $\varepsilon \longrightarrow$



$\rightsquigarrow \varepsilon * u \longrightarrow u$ in $W^{k,p}$ $\lrcorner$

Hölder spaces

Let $0 < \gamma < 1$. We set

$$[u]_{C^{0,\gamma}(\bar{U})} := \sup \left\{ \frac{|u(x)-u(y)|}{|x-y|^\gamma} \right\}$$

&

$$\|u\|_{C^{0,\gamma}(\bar{U})} := \|u\|_{C^0(\bar{U})} + [u]_{C^{0,\gamma}(\bar{U})}$$

Then we define

$$C^{k,\gamma}(\bar{U}) := \left\{ f \in C^k(\bar{U}) \;\middle|\; \|f\|_{C^{k,\gamma}} := \sum_{|\alpha| \leq k} \|D^\alpha f\|_{C^0(\bar{U})} + \sum_{|\alpha| = k} [D^\alpha u]_{C^{0,\gamma}(\bar{U})} < \infty \right\}$$

$C^{k,\gamma}(\bar{U})$ is a Banach space & as such, it is the completion of

$$C^{k,\gamma}(\bar{U}) \cap C^\infty(\bar{U})$$

wrt $\|-\|_{C^{k,\gamma}}$.

Ex / Lemma: If $\varphi: U \to V$ is a diffeomorphism of open subsets of $\mathbb{R}^n$, g is a Riem. metric on V & $\varphi^* g$ is the induced metric on U, get two

Sobolev norms

$$\|f\|_{W^{k,p}(U)} := \sum_{|\alpha| \le k} \left(\int_U |D^\alpha f|^p \, d\mathrm{vol}_g \right)^{1/p}$$

&

$$\|f\|'_{W^{k,p}(U)} := \sum_{|\alpha| \le k} \left(\int_U \left(\frac{\partial^{|\alpha|} (f \circ \varphi)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \right)^p d\mathrm{vol}_{\varphi^* g} \right)^{1/p}.$$

Then if $\overline{U}$ & $\overline{V}$ are compact & φ is the restriction to U of a function on a larger set U' that contains $\overline{U}$, then these two norms are equivalent (e.g. this is true if U & V are charts for a closed manifold) $Y \times \mathbb{R}$, where Y closed: also true

Now, let $f_\alpha : B^n(0) \longrightarrow \mathbb{R}$ be def. by $f_\alpha(x) = \|x\|^\alpha$.

In spherical coordinates

$$\phi : \underbrace{(0,1) \times (0,\pi)^{n-2} \times (0,2\pi)}_{=:\,U} \to B^n$$

the fct. becomes $\begin{pmatrix} r \\ \varphi_1 \\ \vdots \\ \varphi_{n-1} \end{pmatrix} \longmapsto r^\alpha$

& only its partial derivatives $\frac{\partial^k f \circ \phi}{\partial r^k}$

are non-zero.

The volume form $dvol_{\mathbb{R}^n}$ in spherical coordinates is

$$dvol_{\mathbb{R}^n} = r^{n-1} \sin^{n-2}\varphi_1 \cdot \ldots \cdot \sin\varphi_{n-2}\, dr\, d\varphi_1 \cdots d\varphi_{n-1}$$

We know that $f_\alpha \in W^{k,p}(B^n)$

iff $f_\alpha \circ \phi \in W^{k,p}(U)$.

Compute:

$$\left\| \frac{\partial^k (f_\alpha \circ \phi)}{\partial r^k} \right\|_{L^p(U)}^p =$$

$$= \int_{(0,\pi)^{n-2} \times (0,2\pi)} \int_0^1 C \cdot r^{(\alpha-k)p} \cdot r^{n-1} dr \cdot \sin \ldots d\varphi_1 \ldots d\varphi_{n-1}$$

$$= \mathrm{vol}(S^{n-1}) C \cdot \int_0^1 r^{(\alpha-k)p+n-1} dr =$$

$$= \tilde{C} \cdot \left[r^{p(\alpha-k)+n} \right]_0^1 < \infty$$

$$\Leftrightarrow \quad p(\alpha-k)+n \geq 0 \Leftrightarrow \alpha \geq k - \frac{n}{p}$$

Hence $f_\alpha \in W^{k,p}(U)$ iff

$$\alpha \geq l - \frac{n}{p} \quad \text{for all } l \leq k$$

$$\iff \alpha \geq k - \frac{n}{p}$$

Eg. $-\frac{1}{2} \geq 1 - \frac{3}{1}$, so

$$x \longmapsto \frac{1}{\sqrt{\|x\|}}$$

is in $W^{1,1}(B^3)$.

Inequalities:

1) For $u \in C_c^1(\mathbb{R}^n)$, $p \in [1,n)$,

$$p^* := \frac{np}{n-p},$$

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|Du\|_{L^p(\mathbb{R}^n)}$$

(Gagliardo–Nirenberg–Sobolev inequality)

2) For $u \in C^1(\mathbb{R}^n)$, $p \in (n, \infty]$ & $\gamma := 1 - \frac{n}{p}$,

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(\mathbb{R}^n)}$$

(Morrey inequality)

<u>Theorem</u> (Sobolev embedding theorem):

Let $U \subseteq \mathbb{R}^n$ be open, bounded with C^1 boundary & let $u \in W^{k,p}(U)$. Then

i) If $k < \frac{n}{p}$, then $u \in L^q(U)$, where $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$ &

$$\|u\|_{L^q(U)} \leq C \|u\|_{W^{k,p}(U)}$$

$\rightsquigarrow \quad W^{k,p}(U) \hookrightarrow L^q(U)$

bounded.

ii) If $k > \frac{n}{p}$, then $u \in C^{k-\lfloor \frac{n}{p} \rfloor -1, \gamma}$

where $\gamma := \begin{cases} \leq \lfloor \frac{n}{p} \rfloor + 1 - \frac{n}{p}, & \text{if } \frac{n}{p} \notin \mathbb{Z} \\ < 1, & \text{if } \frac{n}{p} \in \mathbb{Z} \end{cases}$

More generally:

<u>Thm</u> (Rellich-Kondrachov compactness thm)

If $k \geq m$ & $k - \frac{n}{p} \geq m - \frac{n}{q}$, then

there is a continuous embedding

$$W^{k,p}(\overline{U}) \subseteq W^{m,q}(\overline{U}).$$

If $\overline{U}$ is compact, & the above inequ. are strict, then the embedding is compact, i.e. if (u_n) is a bounded sequence in $W^{k,p}$, then (u_n) has a $\|-\|_{m,q}$ - convergent subsequence.

Partial Differential Equations by
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