(M^4,ω) closed, symplectic, connected.

$\Longrightarrow$ $\omega^2 =$ Vol Form i.e. ORIENTED.

<u>SETUP</u>: Compatible a.c.s J, $J^2 = -\mathrm{Id}_{TM}$

$\boxed{\omega(\cdot, J_\omega \cdot) = g_J(\cdot,\cdot\cdot)}$ pos. def. METRIC

- $(TM \otimes \mathbb{C})^* = (T^{1,0}M \oplus T^{0,1}M)^*$, $\pm i$-Eigenspaces

$$= \Lambda^{1,0} \oplus \Lambda^{0,1}.$$

$$\begin{tikzcd}
 & & \Lambda^{2,0} \\
 & \Lambda^{1,0} \arrow[ur, "\partial"] \arrow[dr, "\bar\partial"] & \oplus \\
\Omega^0_{\mathbb{C}} = \Lambda^{0,0} \arrow[ur, "\partial"] \arrow[dr, "\bar\partial"'] & \oplus & \Lambda^{1,1} \\
 & \Lambda^{0,1} \arrow[ur] \arrow[dr, "\bar\partial"'] & \\
 & & \Lambda^{0,2}
\end{tikzcd}$$

J integrable $\Longleftrightarrow$ $\boxed{\bar\partial^2 = 0}$ $\Longleftrightarrow$ (M,ω,J) <u>Kähler</u>

<u>Canonical Spin^c-structure</u>:

- $\underline{V_+^{Can}} \oplus V_-^{Can} = \left(\underline{\Lambda^{0,0} \oplus \Lambda^{0,2}}\right) \oplus \Lambda^{0,1}$

If $a \in \Omega^1(M)$, $a \cdot (\alpha,\beta) = \sqrt{2}\left(a^{0,1} \wedge \alpha - *(a^{1,0} \wedge * \beta)\right)$

- $\begin{tikzcd} Spin^c(4) \subseteq U(2) \times U(2) \arrow[r] & SO(4) \\ & U(2) \arrow[u, phantom, "\cup"] \arrow[ul, "\Delta"] \end{tikzcd}$

$$\mathrm{Pic}(M) \longrightarrow \mathrm{Spin}^c(M)$$
$$[L] \longmapsto V^{can} \otimes L$$

- $SW(L) := SW(V^{can} \otimes L)$
- $K_\omega = \Lambda^{2,0}$ Canonical Bundle

Theorem (Taubes): Suppose $b_2^+(M) > 1$, (M,ω) sympl.

Then $\boxed{SW(0) = SW(K_\omega) = \pm 1}$.

~~Rk~~ (More Constraints): If $SW(E) \neq 0$, then

- $0 \leq E \cdot [\omega] \leq K_\omega \cdot [\omega]$

OR

- $E \cdot [\omega] = 0 \wedge K_\omega \cdot [\omega] \Rightarrow E = 0, K_\omega$

Cor: Suppose $b_2^+(M) \geq 2$. Then if

$\underset{SW=0}{\Leftarrow} \begin{cases} \text{(a) } M^4 = M_1 \# M_2, \; b_2^+(M_i) > 0 \\ \text{(b) } M^4 \text{ admits a p.s.c. Metric} \end{cases}$

Then M admits NO symplectic structure.

Ex: Hypersurfaces: $X_n = \{x_0^n + \ldots + x_3^n = 0\} \subseteq (\mathbb{C}P^3, \omega)$

- $\pi_1 X_n = 0$ (LEFSCHETZ HYPERPLANE)
- $X_n \approx p\,\mathbb{C}P^2 \# q\,\overline{\mathbb{C}P}^2$, if $n = 2k+1$ ODD.
(FREEDMAN)

· M Kähler $(\pi_1 M = 1)$

SW-Eqn's: · $D_A^+ \psi = 0$

· $F_A^+ = i\,\sigma(\psi,\psi)$

Use · $\Lambda^2_+ = \underset{\in \Lambda^{1,1}}{\mathbb{R}\cdot\omega} \oplus \Lambda^{0,2}$, $V_+ = \Lambda^{0,0} \oplus \Lambda^{0,2}$ (α, β)

SW-Eqn's (Kähler, V_+^{can} A_0)

$$\boxed{\begin{array}{l} \cdot\ \bar\partial_B \alpha = -\bar\partial_B^* \beta, \quad A = A_0 + B \ \text{(conn. on trivial Bundle)} \\ \cdot\ F^{0,2}_{AB} = \bar\alpha \cdot \beta \\ \cdot\ i F_B^{1,1} = \tfrac{1}{2}(|\beta|^2 - |\alpha|^2)\omega - i F^{11}_{A_0} \end{array}}$$

$$\leadsto \quad \bar\partial_B \bar\partial_B^* \beta = -\bar\partial_B^2 \alpha = -F_B^{0,2} \alpha = -|\alpha|^2 \cdot \beta$$

$$\Rightarrow \quad \int_M \left(|\bar\partial_B^* \beta|^2 + |\alpha|^2 |\beta|^2 \right) dVol = 0$$

$$\Rightarrow \quad \underline{\alpha \equiv 0 \ \&\ \bar\partial_B^* \beta = 0} \ \text{ OR } \ \underline{\beta \equiv 0 \ \&\ \bar\partial_B \alpha \equiv 0}$$

Since $b_2^+(M) \geq 2$, $h^{2,0}(M) \neq 0$ Hodge Nr.

$\Rightarrow$ $h^0(M, \mathcal{K}) \neq 0$. Thus

$$0 \leq -c_1(\overline{\mathcal{K}}) \cdot [\omega] = -\int \frac{i}{2\pi} F_{A_0}^{1,1} \wedge \omega$$

$$= \int_M (|\alpha|^2 - |\beta|^2)\, \omega^2$$

$$\Rightarrow \quad \int_M |\beta|^2 d\mu \leq \int_M |\alpha|^2 d\mu$$

$\Rightarrow$ $\beta \equiv 0$ & $\bar{\partial}_B \alpha \equiv 0$ i.e. α holomorphic

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$\Rightarrow$ unique soln' upto Gauge. ! !

<u>TRICK</u>: PERTURB ~~ω and~~ by (1,1)-part

$$\boxed{i F_A^{1,1} \omega = \tfrac{1}{2}(|\beta|^2 - |\alpha|^2 + \underset{\llcorner}{r})\omega + i F_{A_0}^{1,1}}$$