Chern-Weil theory I

$$\Phi : \mathfrak{g} \longrightarrow \mathbb{R} \quad (\text{or } \mathbb{C})$$

Lie alg of G — polynomial of deg $= k$

alternatively

$$\Phi : \mathfrak{g} \times \dots \times \mathfrak{g} \longrightarrow \mathbb{R}$$

multilin.
invariant under permutations
("symmetric")

suppose ad-invariant:

$$\Phi(\mathrm{ad}_g X_1, \dots, \mathrm{ad}_g X_k)$$
$$= \Phi(X_1, \dots, X_k)$$
$$\forall g \in G$$
$$\forall X_1, \dots, X_k \in \mathfrak{g}$$

Apply this to $g = e^{tX}$
& diff'te at $t=0$

$$\Rightarrow \left[\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0} \Phi(\mathrm{ad}_{e^{tX}} X_1, \dots, \mathrm{ad}_{e^{tX}} X_k) \\ (*) \quad &= \Phi([X, X_1], X_2, \dots, X_k) \\ &\quad + \Phi(X_1, [X, X_2], \dots, X_k) \\ &\quad \dots\dots \end{aligned}\right.$$

Let A be a connection $\begin{array}{c} P \\ \downarrow \\ M \end{array}$

$$c_\Phi(A) := \Phi(\underbrace{\Omega_A \wedge \dots \wedge \Omega_A}_{k \text{ times}})$$

$$\in \Omega^{2k}_{\mathrm{hor}}(P)$$

Ω_A: curvature of A

Two facts from last time:

- $d_A \Omega_A = 0$ Bianchi id.

- $\alpha \in \Omega_{\text{hori},\rho}(P;V)$ $\rho: G \to \mathrm{Aut}\,V$

then

$$(**) \quad d_A \alpha = d\alpha + \underbrace{\rho_*(\omega_A) \wedge \alpha}_{\text{tensorial contraction}}$$

$\rho_* : \mathfrak{g} \to \mathrm{End}\,V$

$(\mathrm{ad})_* = [\,,-]$

<u>Prop^n</u>:

$c_\phi(A)$ is closed, and for any other connection A' on P the difference

$$c_\phi(A) - c_\phi(A')$$ is <u>exact</u>,

hence

$$[c_\phi(A)] \in H^{2k}(P;\mathbb{R})$$

is indep. of A.

Proof:

$$dc_\phi(A) = \phi(d\Omega_A \wedge \Omega_A \wedge \Omega_A \cdots) + \phi(\Omega_A \wedge d\Omega_A \wedge \Omega_A \cdots)$$

$$= k \cdot \phi(d\Omega_A \wedge \Omega_A \wedge \Omega_A \cdots)$$

$$\overset{(*)}{=} k \cdot \phi\big((d\Omega_A + [\omega_A \wedge \Omega_A]) \wedge \Omega_A \wedge \cdots \wedge \Omega_A\big)$$

In fact (*) implies:

$$0 = \phi([\omega_A \wedge \Omega_A] \wedge \Omega_A \wedge \cdots \wedge \Omega_A) + \phi(\Omega_A \wedge [\omega_A \wedge \Omega_A] \wedge \cdots \wedge \Omega_A) + \cdots$$

$$= k \cdot \phi(\overset{=0 \text{ by Bianchi id.}}{d_A \Omega_A} \wedge \Omega_A \wedge \cdots)$$

$$= 0$$

Let A' be another conn.

$$a := A' - A \in \Omega^1_{\text{hori}, \text{ad}}(P; \mathfrak{g})$$

$A_t = A + ta$ is a path of conn. from A to A'.

Then

$$\Omega_{A_t} = \Omega_A + d_A(ta) + \frac{1}{2} t^2 [a \wedge a]$$

$$\Longrightarrow \frac{d}{dt} \Omega_{A_t} = d_A a + t [a \wedge a]$$

$$= d_{A_t} a$$

$$\frac{d}{dt} c_\Phi(A_t)$$

$$= k \cdot \Phi\left(\frac{d\Omega_{A_t}}{dt} \wedge \Omega_{A_t} \wedge \dots \wedge \Omega_{A_t}\right)$$

$$= k \cdot \Phi\left(d_{A_t} a \wedge \Omega_{A_t} \wedge \dots \wedge \Omega_{A_t}\right)$$

__Lemma:__ If $\beta \in \Omega^*_{horiz}(P; \overset{\text{trivial G-action}}{\mathbb{R}})$ G-invariant, then

$$d_A \beta = d\omega$$

__Pf:__ Use $(**)$, with ρ: trivial rep'n. $\square$

$$\underset{\text{identity}}{\overset{\text{Bianchi}}{=}} k \cdot d_{A_t} \Phi\left(a \wedge \Omega_{A_t} \wedge \dots \wedge \Omega_{A_t}\right)$$

$$\overset{\text{Lemma}}{=} k \cdot d\Phi\left(a \wedge \Omega_{A_t} \wedge \dots \wedge \Omega_{A_t}\right)$$

Integration

$$\Longrightarrow \quad c_\Phi(A') - c_\Phi(A) = d\left(k \cdot \int_0^1 \Phi(a \wedge \Omega_{A_\varepsilon} \wedge \dots \wedge \Omega_{A_\varepsilon})\right)$$

$\blacksquare$

Example:

If $\mathfrak{g}$ matrix Lie alg. of matrix Lie group G, then

$$\det(t \cdot \mathrm{Id} + X) =: \sum_{k=0} t^k \Phi_k(X)$$

ad_G-inv. polynomial of degree

(?) $R(G) - k$

Example: $G = U(1)$

$$\mathfrak{g} := \mathfrak{u}(1) = i\mathbb{R}$$

Remark: If $P = M \times G$ trivial bundle, then it admits the trivial connection $pr_1^* TM$, which has 0 curvature (is integrable).

$$\Rightarrow [c_\Phi(\text{triv. conn})] = 0$$

Lemma:

If $\delta : G \to \mathrm{Aut}(V)$ is the trivial hom., then

$$\Omega^*(M; \underbrace{P \times_\delta V}_{\substack{= M \times V \\ \text{triv. bdle}}}) \xrightarrow{\cong} \Omega^*_{\substack{\text{basic} \\ \delta\text{-equiv}}}(P; V)$$

this is given by π^* ☐

Notice that

$$d\pi^* = \pi^* d$$

Therefore $\exists!$ class

$$\check{c}_\phi(A) \in \Omega(M;\mathbb{R})$$

s.t. $\quad \pi^* \check{c}_\phi(A) = c_\phi(A).$

In fact

$$\check{c}_\Phi(A) = \Phi(F_A \wedge \cdots \wedge F_A)$$

where

$$F_A \in \Omega^2(M; ad(P))$$

vector bundle with fibre $\mathfrak{g}$

Example:

$$\begin{tikzcd} & & \mathbb{C}^2 \\ S^1 \arrow[r] & S^3 \arrow[d] \arrow[ur, phantom, "\subseteq"] & (z,w) \arrow[d, mapsto] \\ & \mathbb{CP}^1 \cong S^2 & {[z:w]} \end{tikzcd}$$

Hopf fibration

We will apply the above to

$$c_\phi(A) = -\frac{1}{2\pi i}\,\Omega_A \in \Omega^2(S^3;\mathbb{R})$$

for some conn. A

- ad-action is trivial for $G = S^1$

What is $[\check{c}_\phi(A)] \in H^2_{dR}(S^2;\mathbb{R})$?

de Rham isomorphism:

$$H^2_{dR}(S^2) \longrightarrow \mathbb{R}$$
$$[\omega] \longmapsto \int_{S^2} \omega$$

Chart for $\mathbb{CP}^1$ is

$$\mathbb{C} \xrightarrow{\Phi} \mathbb{CP}^1$$
$$u \longmapsto [u:1]$$

Then $\Phi(\mathbb{C}) = \mathbb{CP}^1 \setminus \{[1:0]\}$

Exercises The:

$$\omega_A = \bar{w}\,dw + \bar{z}\,dz$$

$$\Omega_A = d\omega_A$$

(no quadratic term because $G = S^1$)

$$= d\bar{w} \wedge dw + d\bar{z} \wedge dz$$
$$= - dw \wedge d\bar{w} - dz \wedge d\bar{z}$$

Need to find $F_A \in \Omega^2(S^2; i\mathbb{R})$

s.th. $\pi^* F_A = \Omega_A$

We will express F_A through the dost Φ

In fact:

$$F_A\big|_{\Phi(\mathbb{C})} = s^*\Omega_A$$

because then

$$\pi^* F_A = \pi^* s^* \Omega_A$$
$$= \Omega_A$$

bec. Ω_A is S^1-invariant.

$$\begin{tikzcd} \pi^{-1}(\Phi(\mathbb{C})) \subset S^3 \arrow[d, shift left, "\pi"] \\ \Phi(\mathbb{C}) \arrow[u, bend left, "s"] \end{tikzcd}$$

Looking for a section

$\pi \circ s = id$

$\Rightarrow s^* \circ \pi^* = id$

$$
\begin{tikzcd}
\Phi^*(S^3 \to S^2) \arrow[r, dashed] \arrow[d] & \pi^{-1}(\Phi(\mathbb{C})) \subseteq S^3 \arrow[d] \\
\mathbb{C} \arrow[r, "\Phi"'] & \Phi(\mathbb{C}) \subseteq S^2 \arrow[u, bend left, "s"]
\end{tikzcd}
$$

A candidate is

$$s([u:1]) = \frac{(u,1)}{\sqrt{|u|^2+1}}$$

(defined as

$$(**) \quad u \longmapsto \frac{(u,1)}{\sqrt{|u|^2+1}}$$

composed with Φ^{-1})

$$F_A = s^* \Omega_A$$

$$\Rightarrow \Phi^* F_A = \Phi^* s^* \Omega_A$$

$$= (s \circ \Phi)^* \Omega_A$$

$$(**) = s \circ \Phi$$

$$\Omega_A = -\left(dz \wedge d\bar{z} + dw \wedge d\bar{w}\right)$$

$$\Rightarrow (s \circ \phi)^* \Omega_A$$

$$= -\left(d\left(\frac{u}{\sqrt{|u|^2+1}}\right) \wedge d\left(\frac{\bar{u}}{\sqrt{|u|^2+1}}\right) + 0\right)$$

$$d\left(\frac{u}{\sqrt{|u|^2+1}}\right) = \frac{du}{\sqrt{|u|^2+1}} - \frac{1}{2} u \frac{d|u|^2}{(|u|^2+1)^{3/2}}$$

$$= \underbrace{\frac{du}{\sqrt{|u|^2+1}}}_{1} - \underbrace{\frac{1}{2} u \frac{du \cdot \bar{u} + u\, d\bar{u}}{(|u|^2+1)^{3/2}}}_{2}$$

$$d\left(\frac{\bar{u}}{\sqrt{|u|^2+1}}\right) = \frac{d\bar{u}}{\sqrt{|u|^2+1}} - \frac{1}{2}\bar{u}\left(\frac{du \cdot \bar{u} + u\, d\bar{u}}{(|u|^2+1)^{3/2}}\right)$$

$$= -\left(\overbrace{\frac{du \wedge d\bar{u}}{|u|^2+1}}^{1 \wedge 1} - \frac{1}{2} \overbrace{\frac{|u|^2}{(|u|^2+1)^2} du \wedge d\bar{u}}^{1 \wedge 2}\right.$$

$$\left. - \frac{1}{2} \overbrace{\frac{|u|^2}{(|u|^2+1)^2} du \wedge d\bar{u}}^{2 \wedge 1}\right)$$

$$+\frac{1}{4}\frac{|u|^4}{(|u|^2+1)^3}\,du\wedge d\bar u$$

$$+\frac{1}{4}\frac{u^2\bar u^2}{(|u|^2+1)^3}\,d\bar u\wedge du\Big)$$

$=0$

$$=-\frac{du\wedge d\bar u}{(1+|u|^2)^2}$$

$$(dx+idy)$$
$$\wedge(dx-idy)$$
$$=-2i\,(dx\wedge dy)$$

$$\Rightarrow \int_{\mathbb{CP}^1}\left(-\frac{1}{2\pi i}F_A\right)=\int_{\phi(\mathbb{C})}\left(-\frac{1}{2\pi i}\right)F_A$$

$$=\int_{\mathbb{C}}-\frac{1}{2\pi i}\,\phi^* F_A$$

$$=\int_{\mathbb{C}}\frac{1}{2\pi i}\,\frac{du\wedge d\bar u}{(1+|u|^2)^2}$$

$$= -\int_{\mathbb{C}} \frac{1}{\pi} \frac{dx \wedge dy}{(1+|w|^2)^2}$$

$$= -\frac{1}{\pi} \int_0^{2\pi} \left(\int_0^\infty \frac{r\,dr}{(1+r^2)^2} \right) d\varphi$$

$$= -2 \cdot \int_0^\infty \frac{r}{(1+r^2)^2}\, dr$$

$$= -2 \cdot \left(\frac{1}{2} \left(\frac{-1}{1+r^2} \right) \right) \Big|_0^\infty$$

$$= -1$$

<u>Conclusion:</u>

$$-1 = [c(\text{Hopf-bundle})]$$

$$\in H^2_{dR}(\mathbb{C}P^1)$$

Reduction & Extension of structure group

Def Let $\lambda : H \to G$ be a Lie group homom.
$\pi : P \to M$ a principal G-bundle

A λ-reduction of P is a principal H-bundle $\pi' : Q \to M$ together with a map $f : Q \to P$ satisfying:

$$* \quad \begin{tikzcd} Q \arrow[rr, "f"] \arrow[dr] & & P \arrow[dl] \\ & M & \end{tikzcd}$$

commutative!

$$* \quad f(ph) = f(p)\lambda(h) \qquad \forall p \in Q, \ \forall h \in H$$

$\boxed{* \text{ of type } \lambda}$

Example: $\dim M = n$

$$SO(n) \hookrightarrow GL(n)$$

oriented orthon. frame bundles is a $SO(n)$-reduction of the frame bundle of M

(exists iff TM is orientable)

Rk: P admits a λ-reduction iff $\exists$ cocycles (g_{ik}) coming from cocycles
$$h_{ik}: U_i \cap U_k \to H$$
s.th.
$$g_{ik} = \lambda \circ h_{ik}$$

Example:
$$\lambda: S^1 \to S^1$$
$$z \mapsto z^2$$

Claim: The Hopf bundle $S^3 \downarrow S^2$ does not admit a λ-reduction.

Exercise! (Use Chern classes later on)

* A principal $U(n)$-bundle $P \to M$ admits a reduction to a principal $SU(n)$-bundle iff $P \times_{\det} \mathbb{C}$ is the trivial bundle

Btw: $\boxed{\text{Outlook}}$

- $Spin(n) \xrightarrow{2:1} SO(n)$ (unique double cover)

- Then a $SO(n)$-bundle admits a reduction to a $Spin(n)$-bundle if

$$w_2(P \times_{can} \mathbb{R}^n) = 0$$