Transversality

References

- Moore
- Morgan
- A. Teleman
  Introduction à la théorie de jauge

Setting

(M,g) closed oriented connected Riemannian 4-mfd

Spin^c structure, spinors $S^\pm$
$(P_{\mathrm{Spin}^c(4)} \to M)$ det line bundle L

$\psi \in \Gamma(S^+)$, A: unitary connection on L

configuration space: $\mathcal{A}$

Dirac operator
$$\not{\partial}_A : \Gamma(S^+) \longrightarrow \Gamma(S^-) : \psi \longmapsto e_i \nabla_{A, e_i} \psi$$

$$\not{\partial}_A \psi = 0$$

$$F_A \in \Omega^2(M; i\mathbb{R})$$
$$\Lambda^2_+ \xrightarrow{\cong} su(S^+) \rightsquigarrow i\Lambda^2_+ \xrightarrow{\cong} \mathrm{Herm}_0(S^+)$$

$$F_A^+ = (\psi\psi^*)_0 + i\phi$$

Gauge

$\mathcal{G} \overset{\text{def}}{=} \{M \to S^1\}$ $f \in \mathcal{G} \rightsquigarrow P_{Spin^c(4)} \xrightarrow{\times f} P_{Spin^c(4)}$

$(A,\psi)\cdot f = ((\det f)^* A, S^+(f^{-1})(\psi))$

$= (A + 2f^{-1}df, f^{-1}\psi)$

Configuration (A,ψ)

$\begin{cases} \text{irreducible if stab trivial} & \psi \neq 0 \\ \text{reducible otherwise} & \Longleftrightarrow \psi = 0 \end{cases}$

* Notation

$\Gamma(E), \Omega^k(E)$: C^∞ sections

but sometimes C^ℓ sections for $\ell >> 1$

What we will do

For generic $\phi \in \Omega^2_+(M)$, $\qquad b_2^+ \geq 1$

$$SW : \{\text{Spin}^c\text{-structures}\} \to \mathbb{Z}$$

$$SW(c) = \langle c_1^{\frac{d}{2}}, [\mathcal{M}_{L,\phi}] \rangle$$

$$\left(d = \dim \mathcal{M}_{L,\phi} = \frac{c_1(L)^2 - 2\chi(M) - 3\sigma(M)}{4} \right)$$

c_1 ?

$$\mathcal{G} = \mathcal{G}_0 \times S^1$$

$$\mathcal{A}^*/\mathcal{G}_0 \longrightarrow \mathcal{A}^*/\mathcal{G} = \mathcal{B}^*$$

S^1 bundle $\Rightarrow$ define c_1 $\qquad b_2^+ \geq 2$

does not depend on g and the generic choice of ϕ

1. Need to show that $\mathcal{M}$ cpt mfd

$$f_s : M \to N \qquad s \in S$$

$$F : M \times S \to N$$

$x \in N$ as a regular value

$\Rightarrow$ for generic $s \in S$,

$f_s : M \to N$ has $x \in N$ as a regular value

Problem:

$\mathcal{B} = \mathcal{A}/\mathcal{G}$ is not a manifold

but $b_2^+ \geq 1$

$\exists$ reducibles $\Leftrightarrow i\phi = F_A^+$ for some A

$\Leftrightarrow i\phi = F_{A_0}^+ + id^+\alpha$ for some $\alpha \in \Omega^1$

$$\Omega^1(M) \to \Omega_+^2(M) \to 0$$

Implicit Function Thm

Let X, Y, Z be Banach spaces, $A \subset X \times Y$ open

$f: A \to Z$ C^k-map w/ $f(x_0, y_0) = 0$

If $y \mapsto Df(x_0, y_0)(0, y)$ iso $Y \xrightarrow{\simeq} Z$,

$\rightsquigarrow \exists$ nbds $U \ni x_0$, $V \ni y_0$

& C^k $g: U \to V$ s.t.

$\forall (x, y) \in U \times V$, $f(x, y) = 0 \iff y = g(x)$

Sard-Smale

Let $f: M \to N$ be a C^k-Fredholm map

btw separable Banach mfds

w/ $k > \max(0, \text{Index}(f))$.

Then the set of regular values is residual

3. Non-simply connected

for simply connected M,
used a global Coulomb slice

$$d^*(A - A_0) = 0$$

$k \geq 3 \rightsquigarrow L^2_k \times L^2_k \to L^2_k$

$L^2_k \subset C^0$

$\mathcal{G}_{k+1} = L^2_{k+1}(M, S^1)$ Banach Liegroup

w/ Lie algebra $L^2_{k+1}(M, i\mathbb{R})$

$\mathcal{A}^{(*)}_k = \mathcal{A}_k(L) \times \Omega^0_k(S^+)(\setminus\{0\})$

unitary connections

$\mathcal{A}_k \circlearrowleft \mathcal{G}_{k+1} \quad (A,\psi)f = (A + 2f^{-1}df, f^{-1}\psi)$

$\mathcal{B}^{(*)}_k = \mathcal{A}^{(*)}_k / \mathcal{G}_{k+1}$

$\ell \gg k$

$\phi \in \Omega^2_{+,\ell} \rightsquigarrow \mathcal{M}^{(*)}_{k,\phi}$

Prop $\mathcal{M}_{k,\phi}$ actually does <u>not</u> depend on $k \ll l$

$k_1 \geq k_2$

$$\mathcal{M}_{k_1,\phi} \to \mathcal{M}_{k_2,\phi}$$

any L^2_k-solution is L^2_l

Recall A A_0 $U(1)$-connection

$\exists$ gauge tr f s.t. $A' = A \cdot f$ $d^*(A' - A_0) = 0$

$$A'' = A' + 2f^{-1}df \qquad 2\pi i H^1(M, \mathbb{Z})$$

$$d^*(\underbrace{f^{-1}df}_{\alpha}) = 0$$

$$\{f^{-1}df : f : M \to S^1, d^*(f^{-1}df) = 0\}$$

$$= 2\pi i \underset{\substack{\cap \\ \text{harmonic}}}{H^1}(M, \mathbb{Z})$$

$\Rightarrow$ $\exists f : M \to S^1$ s.t.

$A' = A \cdot f$ satisfies

$$d^*(A' - A_0) \in \Pi \leftarrow \text{compact}$$

$\uparrow$ fundamental domain of the lattice $2\pi i\, H^1(M, \mathbb{Z})$ in $H^1(M, \mathbb{R})$

For any $A = A_0 + \alpha$ $\quad \alpha \in i\Omega^1(M)$ & $\psi \in \Gamma(S^+)$,

wLOG after gauge tr,

$$\begin{cases} \not{D}_{A_0}\psi = -\frac{1}{2}\alpha \cdot \psi \\ \begin{pmatrix} d^+ \\ d^* \end{pmatrix}\alpha = \begin{pmatrix} (\psi\psi^*)_0 - F^+_{A_0} \\ 0 \end{pmatrix} \end{cases}$$

$\Delta\alpha \in \Pi$

ψ $\quad C^0$ bdd

$\begin{pmatrix} d^+ \\ d^* \end{pmatrix}\alpha$ $\quad C^0$ bdd

$\Delta\alpha = \mathrm{pr}^{L^2}_{\ker(d^+_\delta)}\, \alpha$ bdd

$\Rightarrow \alpha \quad L^p_1$ bdd $\quad L^2_k$

$\Rightarrow -\frac{1}{2}\alpha \cdot \psi \quad L^p$ bdd

$\Rightarrow \psi \quad L^p_1$ bdd

$\alpha, \psi \in C^\infty$

$\mathcal{M}^*$

$$0 \to i\Omega^0(M) \xrightarrow{D_p^0} \overbrace{i\Omega^1(M) \oplus \Omega^0(S^+)}^{C^1} \xrightarrow{D_p^1} i\Omega^2_+(M) \oplus \Omega^0(S^-) \to 0$$

for $p = (A, \psi) \in \mathcal{A}_\alpha^k$

define

$$o_p : \mathcal{G}_{k+1} \to \mathcal{A}_k^* : f \mapsto p \cdot f$$

$$D_p^0(g) = \begin{pmatrix} 2dg \\ -g \cdot \psi \end{pmatrix}$$

$D o_p|_e$

$$D_p^1 \begin{pmatrix} \alpha \\ \eta \end{pmatrix} = \begin{pmatrix} d^+\alpha - (\psi\eta^*)_0 - (\eta\psi^*)_0 \\ \not{D}_A \eta + \frac{1}{2}\alpha \cdot \psi \end{pmatrix}$$

$D_p^1 \oplus D_p^{0*}$ is elliptic

if $\ker D_p^0 = 0$, $\operatorname{coker} D_p^1 = 0$,
then the tangent space of $\mathcal{M}^*$
will be $\ker(D_p^1 \oplus D_p^{0*})$

$$C^1_k = \ker D_p^{0*} \oplus \operatorname{Im} D_p^0$$

$$S_p^\varepsilon = \{p + \alpha : D_p^{0*}(\alpha) = 0,\ \|\alpha\|_{L^2_k} < \varepsilon\},$$

then

<u>lemma</u> For $\varepsilon > 0$ suff. small,

$$S_p^\varepsilon : S_p^\varepsilon \times \mathcal{G}_{k+1} \to \mathcal{A}_k^*$$

is a diffeo onto an open submfd of $\mathcal{A}_k^*$

<u>Thm</u> $\mathcal{B}_k^*$ is a Banach mfd, $\pi : \mathcal{A}_k^* \to \mathcal{B}_k^*$ is a principal $\mathcal{G}_{k+1}$-bundle

$\mathcal{M}_\phi$

$$\begin{tikzcd} & \mathcal{A}_k^* \times_{\mathcal{G}} C \arrow[d] \\ & \mathcal{B}_k^* \arrow[u, dashed, bend left, "sw"] \end{tikzcd}$$

$$0 \to i\Omega^0(M) \xrightarrow{2d} i\Omega^1(M) \xrightarrow{d^+} i\Omega^2_+(M) \to 0$$

$$\oplus \qquad\qquad \oplus$$

$$0 \to \Omega^0(S^+) \xrightarrow{\not{D}_A} \Omega^0(S^-) \to 0$$

<u>Thm</u> $\mathrm{Ind}(\not{D}_A) = \dfrac{c_1^2 - \sigma(M)}{4}$

and so

$\mathrm{Ind}(D_p^1 \oplus D_p^{0*})$

$$= \frac{c_1^2 - \sigma(M)}{4} + b_1 - b_2^+ - b_0$$

$$= \frac{c_1^2 - 2\chi(M) - 3\sigma(M)}{4}$$

$f_0 : M \to N$

$F : M \times S \to N$

$$\begin{array}{c} X \subset M \times S \\ \pi \big\downarrow \\ S \end{array}$$

$$SW : \mathcal{A}^*_k \times \Omega^2_{+,\ell}(M) \to \overbrace{i\Omega^2_{+,k-1}(M) \oplus \Omega^0_{k-1}(S)}^{C}$$

$$\begin{array}{c} \mathcal{A}^*_k \times \Omega^2_{+,\ell}(M) \times_{\mathcal{G}_{k+1}} C \\ \big\downarrow \\ \mathcal{B}^*_k \times \Omega^2_{+,\ell}(M) \end{array}$$

SW

pf of that $SW \pitchfork 0$ section.

$q_0 = (p_0 = (A_0, \psi_0), \phi_0)$ is a soln

WTS: $D_{(q_0)} SW$ is surjective,

suff: differential from

$\mathcal{A}^*_k \times \Omega^2_{+,\ell}(M)$ is surjective @ q_0

suff to show that

the L^2-complement of

$\operatorname{Im} d_{q_0} SW$ is 0

$(\chi, \eta) \in i\Omega^2_{+,k-1}(M) \oplus \Omega^0_{k-1}(S^-)$

is L^2-orthogonal to Im

① $\chi = 0$

because

$\operatorname{Im}(SW|_{0 \oplus \Omega^2_{+,l}(M)} \quad \Omega^2_{+,l}(M) \to i\Omega^2_{+,k-1}(M))$

is $i\Omega^2_{+,l}(M)$ (so dense in $i\Omega^2_{+,k-1}(M)$)

$\chi \perp i\Omega^2_{+,l}(M)$

② $\forall \alpha \in i\Omega^1(M)$,

$\langle \frac{1}{2}\alpha \cdot \psi_0, \eta \rangle_{L^2} = 0$

For pts $x \in M$ s.t.

$\psi_0(x) \neq 0$

$$i\Lambda^1(M) \xrightarrow{\sim} S^-$$
$$\alpha_x \longmapsto \frac{1}{2}\alpha_x \cdot \psi_0(x)$$

$\eta(x) = 0$

but $\psi_0 \not\equiv 0$

since $\not{D}_A \psi_0 = 0$, and $\not{D}_A$ is elliptic,

by the unique continuation principle,

$\{x : \psi_0(x) \neq 0\}$ is dense

$\Rightarrow \eta = 0$ on a dense set

$\Rightarrow \eta = 0$

$\mathcal{M} = Z(SW)$
$\downarrow \pi$
$\Omega^2_{+,\ell}(M)$

Claim: π is Fredholm

and so generic $\phi \in \Omega^2_{+,\ell}(M)$ is a regular value

For such ϕ,

$\mathcal{M}^*_\phi$ is a manifold

and if perturbing by ϕ does not give any reducibles (codim b_2^+-cond) ($b_2^+ > 1$)

then $\mathcal{M}^*_\phi = \mathcal{M}_\phi$ is a cpt manifold

$b_2^t \leq 2$

$\mathcal{M}_{\phi_0}$ $\mathcal{M}_{\phi_1}$

Thank you

for

listening!