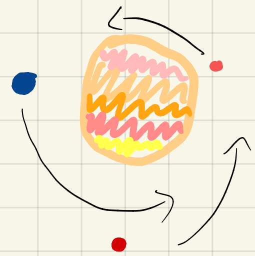
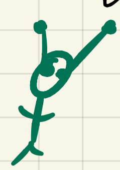
Seiberg Witten: compactness

Compactness of the moduli-space of solutions to the Seiberg-Witten Equations*





*(for simply connected M)

SW-equations (perturbed)

(M,g) closed oriented Riemannian manifold

$\alpha : \Gamma(W_+ \otimes L) \longrightarrow \Omega^2_+(M)$

quadratic!

$\alpha(\psi) = -\frac{1}{2} \sum_{i<j} \langle \psi, e_i \cdot e_j \cdot \psi \rangle e_i \cdot e_j$

$|\alpha(\psi)| = \frac{1}{\sqrt{2}} |\psi|^2$

$$D_A^+ \psi = 0, \qquad F_A^+ = \alpha(\psi) + \phi$$

$\phi \in \Omega^2_+(M)$

$\psi \in \Gamma(W_+ \otimes L)$

$D_A^+ : \Gamma(W_+ \otimes L) \longrightarrow \Gamma(W_- \otimes L)$

roughly: (Clifford multiplication) $\circ$ (Spin^c - connection)

$D_A^+(\psi) = \sum_i e_i \cdot \nabla^A_{e_i} \psi$

I) Brief Introduction to / recapitulation of basic facts in

Hodge Theory $(\dim = 4)$

(M,g) closed oriented Riemannian manifold, $\{e^i\}$ (local) orthonormal frame,
pos. oriented

<u>define:</u> •) $* : \Omega^k(M) \longrightarrow \Omega^{4-k}(M)$

$$e^{i_1}\wedge\dots\wedge e^{i_k} \longmapsto e^{i_{k+1}}\wedge\dots\wedge e^{i_4} \qquad \text{if } e^{i_1}\wedge\dots\wedge e^{i_4} > 0$$

extend C^∞-linearly to bundle isomorphism $\Lambda^k T^*M \to \Lambda^{n-k} T^*M$

e.g.: $*1 = e^1\wedge e^2\wedge e^3\wedge e^4$ $\qquad * e^1\wedge e^2 = e^3\wedge e^4$

$*e^1 = e^2\wedge e^3\wedge e^4$ $\qquad$ etc

$*^2 = (-1)^k$ on k-forms,

•) $\langle -,- \rangle : \Omega^k(M)\times\Omega^k(M) \longrightarrow \mathbb{R}$

$$\langle \omega, \theta \rangle = \int_M \underbrace{\omega \wedge *\theta}_{\in \Omega^n(M)}$$

symmetric, bilinear & positive definite!

•) $\delta : \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$

$$\delta = - * \circ d \circ *$$

$$\begin{tikzcd}
\Omega^{n-k+1} \arrow[dddd, bend left=60, "-*"] \\
\Omega^{n-k} \arrow[u, "d"] \\
\vdots \\
\Omega^{k} \arrow[uu, bend right=60, "*"'] \arrow[d, "\delta"] \\
\Omega^{k-1}
\end{tikzcd}$$

$\omega \in \Omega^k(M)$

$$\langle d\omega, \theta \rangle = \int_M d\omega\wedge *\theta = \int_M d(\omega\wedge *\theta) - (-1)^k \int_M \omega \wedge d*\theta$$

$$= -(-1)^k (-1)^{4-k} (-1) \int_M \omega\wedge *\delta\theta = \langle \omega, \delta\theta \rangle$$

$\cdot)\ \Delta = \delta d + d\delta : \Omega^k(M) \longrightarrow \Omega^k(M)$

"Hodge – Laplacian"

if $\Delta\omega = 0$, ω is called harmonic

call $\mathcal{H}^k(M) := \{\omega \in \Omega^k(M) \mid \Delta\omega = 0\}$

$f \in C^\infty(M) =: \Omega^0(M)$

Δf = geom. Laplacian of f

$\mathcal{H}^0 = \{\text{constant functions}\}$

$\dim \mathcal{H}^0 = \dim H^0(M;\mathbb{R}) = 1$ ✓

<u>Theorem</u> (Hodge):

$H^k(M;\mathbb{R}) \cong \mathcal{H}^k(M)$

& $\Omega^k(M) = \mathcal{H}^k(M) \oplus \Delta(\Omega^k(M)) = \mathcal{H}^k(M) \oplus d(\Omega^{k-1}(M)) \oplus \delta(\Omega^{k+1}(M))$

orthogonal wrt $\langle -,- \rangle$

application: $*\Delta = \Delta *$, so $*: \mathcal{H}^k \longrightarrow \mathcal{H}^{n-k}$ isomorphism

$\Rightarrow H^k(M;\mathbb{R}) \cong H^{n-k}(M;\mathbb{R})$

since $*^2 = 1$ on $\Omega^2(M)$, splitting $\Omega^2(M) = \underset{*\omega = \omega}{\Omega^2_+(M)} \oplus \underset{*\omega = -\omega}{\Omega^2_-(M)}$

self dual — anti-self dual

get projectors $P_\pm(\omega) = \frac{1}{2}(\omega \pm *\omega)$

$\omega^+ := P_+(\omega) \qquad d^+ := P_+ \circ d$

since $[*, \Delta] = 0$, $[P_\pm, \Delta] = 0$, so get splitting

$\mathcal{H}^2(M) \cong \mathcal{H}^2_+(M) \oplus \mathcal{H}^2_-(M)$

$b_+ = \dim \mathcal{H}^2_+ \qquad b_- = \dim \mathcal{H}^2_- \qquad b_2 = b_+ + b_-$

$\tau(M) = b_+ - b_-$ ... signature of M

Fundamental elliptic complex

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d^+} \Omega^2_+(M) \to 0$$ is exact

in particular:

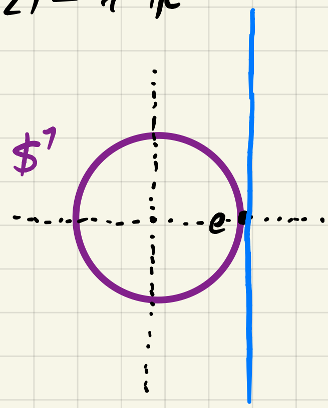
$$\boxed{\ker d^+ = \operatorname{Im} d}$$

M simply connected

II) The Moduli Space

recall: .) connections on $G-VB$ are affine copy of $\Omega^1(M,\mathfrak{g})$

.) L is $U(1) \cong SO(2)$ bundle, so connection 1-forms take values in $so(2) \cong i\cdot\mathbb{R}$



$T_e SO(2) = so(2) \cong i\cdot\mathbb{R}$

so connection 1-forms $\Omega^1(M, i\cdot\mathbb{R}) = i\cdot\Omega^1(M,\mathbb{R})$

We choose a "base connection" d_{A_0} on L & define

$$\mathcal{A} = \{ (d_{A_0} - ia, \psi) \mid a \in \Omega^1(M),\ \psi \in \Gamma(W_+ \otimes L) \}$$

the configuration space.

$$\begin{tikzcd} U(1) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U(1) \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$

Now $\mathcal{G} := \{ g : M \to S^1 \text{ smooth} \}$ is the Group of Gauge transformations

($U(1)$-bundle automorphisms covering the identity)

$$\begin{aligned} \mathcal{G} \times \mathcal{A} &\longrightarrow \mathcal{A} \\ (g, (d_{A_0} - ia, \psi)) &\longmapsto (d_{A_0} - ia + g\,d(g^{-1}), g\cdot\psi) \end{aligned}$$

find global logarithm of any $g \in \mathcal{G}$:

$$\begin{array}{ccc} & & \mathbb{R} \\ & \overset{u}{\nearrow} & \big\downarrow p = e^{i(\cdot)} \\ g: M & \longrightarrow & S^1 \end{array}$$

M simply connected

$g_* \pi_1(M) = e \subseteq p_* \pi_1 \mathbb{R}$

$p \circ u = e^{iu} = g$

lift u ($=$ logarithm $p(u) = e^{iu} = g$) exists for all g

$\Rightarrow$ action $\mathcal{G} \times \mathcal{A} \longrightarrow \mathcal{A}$ takes form

$$\boxed{(g, (d_{A_0} - ia, \psi)) \mapsto (d_{A_0} - ia + e^{iu} d e^{-iu}, e^{iu}\psi) = (d_{A_0} - i(a + du), e^{iu}\psi)}$$

does not see constant shifts i.e. $d(u + \text{const}) = du$

$\mathcal{G}_0 := \{ g \in \mathcal{G} \mid g(p_0) = 1 \} \leq \mathcal{G}$ for some fixed $p_0 \in M$

"the group of based Gauge transformations"

$\mathcal{G}_0$ acts freely on $\mathcal{A}$!

$\tilde{\mathcal{B}} := \mathcal{A} / \mathcal{G}_0$

$\mathcal{G} = \mathcal{G}_0 \times U(1)$

constant Gauge transformations

Lemma (Gauge fixing)

If M is simply connected then each element of $\tilde{B}$ has a unique representative of the form

$$(d_{A_0} - ia, \psi) \quad \text{where} \quad \delta a = 0$$

Pf:

existence: enough to find function $u \in \Omega^0(M)$ s.t.

$$\delta(a+du) = 0 \qquad \delta a \in \Omega^0(M)$$

$$\Leftrightarrow \quad \Delta u = d\delta u + \delta d u = \delta d u = -\delta a \qquad \text{(Poisson's equation)}$$

$$\langle 1, \delta a \rangle = \langle d1, a \rangle = 0 \Rightarrow \delta a \perp \mathcal{H}^0(M), \text{ harmonic functions (i.e. the constant ones)}$$

Hodge: $\Omega^0(M) = \mathcal{H}^0(M) \oplus \Delta\Omega^0(M)$

$$\Rightarrow \delta a \in \Delta\Omega^0(M) \Leftrightarrow \delta a = \Delta u \text{ for some } u \in \Omega^0(M)$$

✓

uniqueness: if $(d_{A_0} - ia_1, \psi_1)$ & $(d_{A_0} - ia_2, \psi_2)$ are gauge equivalent & $\delta a_1 = 0 = \delta a_2$

then $\exists f \in \Omega^0(M)$ s.t. $a_1 = a_2 + df$

$$\Rightarrow da_1 = da_2$$

M simply connected

$$\Rightarrow a_1 - a_2 = du$$

$$\|a_1 - a_2\|^2 = \langle a_1 - a_2, a_1 - a_2 \rangle = \langle du, a_1 - a_2 \rangle$$

$$= \langle u, \underbrace{\delta(a_1 - a_2)} \rangle = 0$$

$$\Rightarrow a_1 = a_2$$

✓ $\square$

Consequence:

$$\tilde{B} \xleftrightarrow{1:1} \{(d_{A_0} - ia, \psi) \mid a \in \Omega^1(M), \psi \in \Gamma(W_+ \otimes L), \delta a = 0\}$$

$\cap$ vector subspace

$\mathcal{A}$

actually want to divide out full Gauge group

$\lightning$ Problem: $\mathcal{G}/\mathcal{G}_0 \cong U(1)$ acts freely on $\tilde{\mathcal{B}}$ except on elements where $\psi = 0$.

quotient space $\mathcal{B} := \mathcal{A}/\mathcal{G}$ has singularities at "reducible" elements $(d_{A_0} - ia, 0)$

[can excise these reducible elements

$$\mathcal{A}^* := \mathcal{A} \setminus \{(d_A - ia, 0)\}_{a \in \Omega^1(M)} \rightsquigarrow \text{next talk(s)}$$

$$\tilde{\mathcal{B}}^* := \mathcal{A}^*/\mathcal{G}_0 \qquad \mathcal{B}^* := \mathcal{A}^*/\mathcal{G}$$
]

Def: (monopole moduli space)

$$\mathcal{M} = \{[A,\psi] \in \mathcal{B} = \mathcal{A}/\mathcal{G} \mid (A,\psi) \text{ satisfies SW}\}$$

$$\mathcal{M}_\phi = \{[A,\psi] \in \mathcal{B} \mid (A,\psi) \text{ satisfies } \phi\text{-perturbed SW}\}$$

using the Gauge-fixing Lemma we see that $\mathcal{M}_\phi$ is the quotient of

$$\tilde{\mathcal{M}}_\phi = \left\{ (d_{A_0} - ia, \psi) \in \Omega^1(M) \times \Gamma(W_+ \otimes L) \;\middle|\; \begin{array}{l} (d_{A_0} - ia, \psi) \text{ satisfies } \phi\text{-perturbed SW}, \\ \delta a = 0 \end{array} \right\}$$

by the action of $S^1 \cong U(1)$

III) Compactness of $\mathcal{M}_\phi$

Turn to topology

Given E a smooth $O(n)$ or $U(n)$ VB over M with connection on it

$\rightsquigarrow$ use LC connection to define connection d_A on $\bigotimes^k T^*M \otimes E$

for $s \in \Gamma(E)$: $\quad d_A^k s := \underbrace{(d_A \circ \ldots \circ d_A)}_{k} s \in \Gamma\left(\bigotimes_k T^*M \otimes E\right)$

$$p > 1 \qquad \|s\|_{p,k} = \left(\int_M \left[|s|^p + |d_A s|^p + \ldots + |d_A^k s|^p \right] dV \right)^{1/p}$$

This is a norm on $\Gamma(E)$!!

unproven claim: different choices of Riemannian metric, fiber metric on E, and connection on E lead to equivalent norms !

Now complete $\Gamma(E)$ wrt $\|-\|_{p,k}$, call this space $L^p_k(E)$

This clearly is a Banach space for all p & Hilbert for $p=2$.

List of key facts from functional analysis

($\dim M = 4$)

1) Sobolev embedding Thm:

If $k - 4/p > \ell$ then there is a continuous embedding

$$L^p_k(E) \longrightarrow C^\ell(E) \qquad C^\ell \text{ sections of } E$$

2) Rellich's Thm:

The inclusion $L^p_{k+1}(E) \hookrightarrow L^p_k(E)$ is compact $\forall\, p, k$

3) Sobolev multiplication Thm:

Suppose that $k - 4/p > 0$. Then the multiplications

$$L^p_k(E) \times L^p_k(F) \longrightarrow L^p_k(E \otimes F) \qquad \text{are continuous}$$

We say that (p,k) are in Banach algebra range

★) compact operators are an "ideal" in bounded operators

$$X \xrightarrow{A} Y \xrightarrow{C} Z \xrightarrow{B} W \qquad \begin{array}{l} A, B \text{ bounded} \\ C \text{ compact} \end{array}$$

Then $A \circ C$ & $C \circ B$ are compact

Lemma: If (A,ψ) is a solution to SW with ψ not identically zero, and the maximum value of $|\psi|$ is assumed at a point $p \in M$, then

$$|\psi|^2(p) \leq -\frac{1}{4} s(p)$$

where s is the scalar curvature.

Moore Lemma 23, p. 79

ingredients:

- use that $\Delta |\psi|^2(p) \geq 0$

- use SW I and II & Weitzenböcks formula
$$\left(D_A^2 \psi = \Delta^A \psi + \frac{s}{4}\psi - \sum_{i<j} F_A(e_i,e_j)(i e_i \cdot e_j \cdot \psi)\right)$$

Interpretation: solutions to SW are bounded!

(M,g) pos. scalar curvature $\Rightarrow$ no solutions

Key inequality: If D is a Dirac operator or (Dirac operator with coefficients) then

$$\|s\|_{p,k+1} \leq c\left(\|Ds\|_{p,k} + \|s\|_{p,k}\right) \qquad c \in \mathbb{R}$$

If D has trivial kernel then the $\|s\|_{p,k}$ term may be omitted

Claim: •) $(\delta \oplus d^+)$ is a Dirac operator (with cofficients in W^+)

•) on M simply connected $\delta \oplus d^+ : \Omega^1(M) \longrightarrow \Omega^0(M) \oplus \Omega^2_+(M)$ (o)
has trivial kernel

proof: suppose $(\delta\alpha, d^+\alpha) = (0,0)$

by exactness of fundamental elliptic complex,
$\operatorname{Im} d = \ker d^+$, so $\alpha = df$, $\quad f \in \Omega^0(M)$

but $0 = \delta\alpha = \delta df = (\delta d + d\delta) f = \Delta f$
so $f \in \mathcal{H}^0(M) \Rightarrow f$ constant $\Rightarrow \alpha = df = 0$ $\square$

Compactness Thm: If M is simply connected, then for any choice of self-dual 2-form ϕ, the moduli space $\tilde{\mathcal{M}}_\phi$ of solutions to the SW equations is compact.

proof: need to show: any sequence $[d_{A_0} - i a_j, \psi_j]$ of solutions to SW possesses a convergent subsequence (in C^∞)

having imposed the Gauge condition $\delta a = 0$ makes the SW equations look like this

$$\left\{\begin{aligned} D_{A_0}\psi - i a\cdot\psi &= 0 \\ (da)^+ + F_{A_0}^+ &= \sigma(\psi) + \phi \\ \delta a &= 0 \end{aligned}\right\} \iff \left\{\begin{aligned} D_{A_0}\psi &= i a\cdot\psi \\ (da)^+ &= \sigma(\psi) + \phi - F_{A_0}^+ \\ \delta a &= 0 \end{aligned}\right\}$$

LHS: 2 Dirac operators!

By Lemma: ψ_j bounded in $C^0 \Rightarrow \psi_j$ bounded in L^p $\forall p$

by (0): $\|a_j\|_{p,1} \leq c\cdot \|(\delta \oplus d^+)a_j\|_{p,0} \leq c\cdot\left(\|\sigma(\psi_j)\|_{p,0} + \|\phi\|_{p,0} + \|F_{A_0}^+\|_{p,0}\right) < \tilde{C}$

$\Rightarrow a_j \in L_1^p$ $\forall p$ $\Rightarrow$ bounded in C^0 by Sobolev embedding

$\Rightarrow$ $a_j\psi_j$ bounded in L^p $\forall p$

$\Rightarrow$ $\|\psi_j\|_{p,1} \leq c\cdot\left(\|D_{A_0}\psi_j\|_{p,0} + \|\psi_j\|_{p,0}\right) \leq \tilde{c}\,\|a_j\psi_j\|_{p,0} < \hat{C}$

now we bootstrap: a_j, ψ_j bounded in $L_1^p \Rightarrow a_j\psi_j, \sigma(\psi_j)$ bounded in L_1^p

$\Rightarrow a_j, \psi_j$ bounded in L_2^p

in general: $\|a_j\|_{p,k+1} \leq c\,\|(\delta\oplus d^+)a_j\|_{p,k} \leq \tilde{c}\,\|\psi_j\|_{p,k}^2$

$\|\psi_j\|_{p,k+1} \leq \quad \ldots\ldots \quad \leq \tilde{c}\,\|a_j\psi_j\|_{p,k}$

$\Rightarrow$ a_j, ψ_j bounded in L^p_k $\forall p,k$

$\Rightarrow$ by Rellich they have a subsequence that converges in L^p_k $\forall p,k$

pf:

$$\begin{array}{l} \qquad\vdots \\ L^p_3 \supset \quad u^1_n \\ L^p_2 \supset u^0_n \\ L^p_1 \supset u^0_n \supset u^1_n \\ L^p_0 \supset \quad u^0_n \end{array}$$

u^1_n convergent subsequence

get sequence of convergent sequences u^k_n in L^p_k

$\rightsquigarrow$ take diagonal sequence

$\rightsquigarrow$ have sequence that converges in L^p_k $\forall p,k$ $\square$

$\Rightarrow$ by Sobolev embedding this subsequence converges in C^ℓ $\forall \ell$.

$\square$

Thank You

for Your

Attention

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