Adjunction inequality (and Thurston norm)

I. The 3-dim'l SW equs

II. The apriori bound

III. The stretching argument

IV. Appl. 1: Adjunction ineq.

$$\boxed{\begin{array}{l}\text{If } SW_X(\mathfrak{s}) \neq 0 \text{ then} \\ |\langle c_1(\mathfrak{s}), [\Sigma] \rangle| + \Sigma \cdot \Sigma \leq \chi_-(\Sigma) \\ \text{if } \Sigma \cdot \Sigma \geq 0. \qquad \text{for } \Sigma \hookrightarrow X\end{array}}$$

where $\chi_-(\Sigma) = \sum_i \max(0, -\chi(\Sigma_i))$

where $\Sigma = \coprod \Sigma_i$

$= 2g(\Sigma)$

I.

$$\gamma : T^*Y \longrightarrow \mathrm{End}_0(W)$$

W: herm. rk $= 2$ bdle

Clif. mult.

$$\boxed{\begin{aligned} U(2) &= \\ Spin^c(3) &= SU(2)\times S^1/\mathbb{Z}/2 \\ &= Spin(3)\times S^1/\mathbb{Z}/2 \end{aligned}}$$

$Spin^c(3)$-connect. conn. on W s.th. γ is parallel w.r.t. Levi-Civita connection on T^*Y

Indeterm.

Choice of conn on trace part of $\mathrm{End}(W)$ = conn. on $\Lambda^2 W$ or $\det(W)$

$\mathfrak{s} = (W_{\mathfrak{s}}, \rho)$

$c_1(\mathfrak{s}) = c_1(\det(W))$

A a $Spin^c(\mathfrak{s})$-conn.

$\rightsquigarrow \quad \Gamma(W) \xrightarrow{\nabla_A} \Gamma(T^*M \otimes W)$

$\xrightarrow{\rho} \Gamma(W)$

$\mathcal{D}_A := \rho \circ \nabla_A$

The SW$_{\mathfrak{s}}$-equs are:

$$\boxed{\begin{aligned} \rho(F_{\hat{A}}) &= (\Psi\Psi^*)_0 \\ D_A \Psi &= 0 \end{aligned}}$$

$\hat{A}$ induced conn in $\Lambda^2 W = \det(W)$

II. Weitzenböck formula:

$$D_A \circ D_A = \nabla_A^* \nabla_A + \frac{s}{4} + \frac{1}{2}\rho(F_A)$$

Lemma Suppose (A,Φ) solves the SW equs on a cpt, closed 3-mfld. Then

$$|\Phi|^2 \le \max(0, \max(-s))$$

Pf:

$$\Delta|\Phi|^2 = 2\langle \nabla_A^*\nabla_A\Phi, \Phi\rangle \underbrace{-2\langle \nabla_A\Phi, \nabla_A\Phi\rangle}_{\le 0}$$

$$\overset{\text{Weitz. \& SW-equs}}{\le} -\frac{s}{2}|\Phi|^2 - \langle \rho(F_A)\Phi, \Phi\rangle$$

$$\overset{\text{SW equ}}{=} -\frac{s}{2}|\Phi|^2 - \underbrace{\langle (\Phi\Phi^*)_0\Phi, \Phi\rangle}_{\overset{(*)}{=}\frac{1}{2}|\Phi|^4}$$

$$= -\frac{s}{2}|\Phi|^2 - \frac{1}{2}|\Phi|^4$$

Q*1: In a basis $\left(\frac{\Phi}{|\Phi|}, \frac{\Phi^\perp}{|\Phi|}\right)$

$$\left(\Phi\Phi^*\right)_0 = \begin{pmatrix} |\Phi|^2 & 0 \\ 0 & 0 \end{pmatrix}_0$$

$$= \begin{pmatrix} \frac{1}{2}|\Phi|^2 & 0 \\ 0 & -\frac{1}{2}|\Phi|^2 \end{pmatrix}$$

Now at a pt x_0 where $|\Phi|^2$ becomes max

$$\left(\Delta |\Phi|^2\right)_{x_0} \geq 0$$

Convention: $\Delta = -\sum \frac{\partial^2}{\partial x_i^2}$

$\Rightarrow$

$$0 \leq -\frac{s}{2}|\Phi|^2 - \frac{1}{4}|\Phi|^4 \text{ at } x_0$$

$\Phi(x_0) \neq 0$ $\Rightarrow$

$$\boxed{|\Phi|^2(x_0) \leq -8(x_0)}$$

$\blacksquare$

III.

Def^{n} A class $\alpha \in H^2(Y^3;\mathbb{Z})$ is said to be a **monopole class** if $\alpha = c_1(\mathfrak{s}_Y)$ and SW_3 equ. for $\mathfrak{s}_Y$ have solutions for any choice of Riem. metric.

Recall: A class $\alpha \in H^2(X^4;\mathbb{Z})$ is a **basic class** if $SW_X(\mathfrak{s}) \neq 0$ where $\alpha = c_1(\mathfrak{s})$.

Theorem (Stretching argument)

Let Y be a closed oriented 3-mfd embedded in a closed oriented 4-mfd X. If $\alpha \in H^2(X;\mathbb{Z})$ is a basic class, then $\alpha|_Y$ is a monopole class.

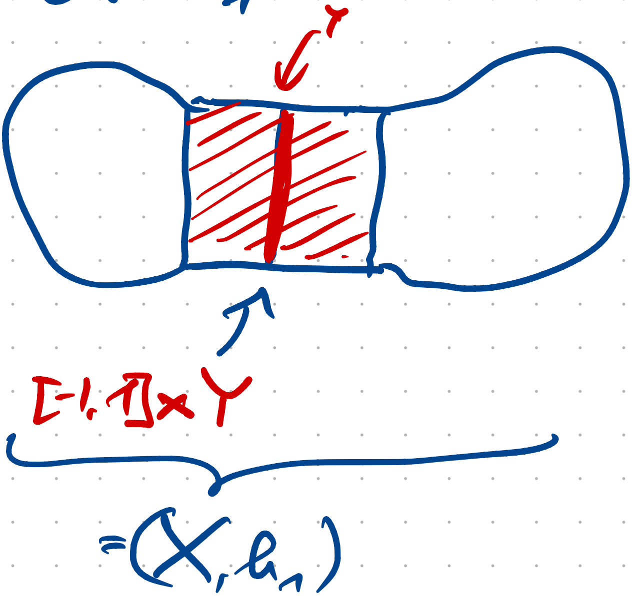
Proof sketch: Let $SW_X(s) \neq 0$

$\Rightarrow$ There are solutions of SW_X eqn $\forall$ Riem. metrics on X.

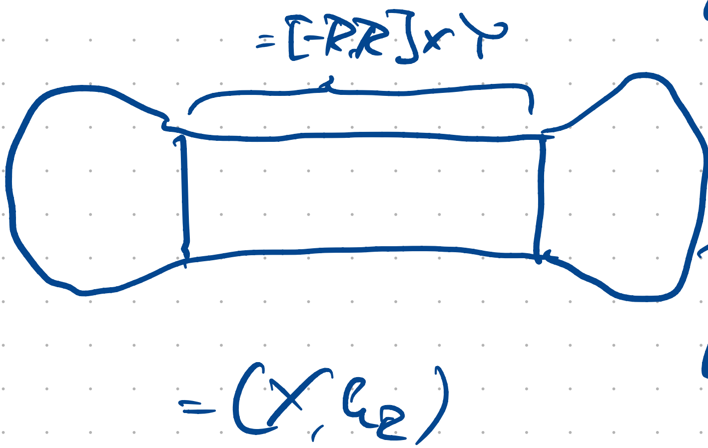
Let h_Y be a Riem. metric on Y.

Let h_1 be a metric which is the product metric on $[-1,1]\times Y$

$$\boxed{dt^2 + h_Y}$$



Let h_R for $R>1$ be equal to h_1 outside of $[-1,1]\times Y \subseteq X$, but the neck is isometric to $[-R,R]\times Y$



By assumption $\exists$ soln (Φ_R, A_R) on X for any $R>1$.

$\mathfrak{s} = (W_s^+, W_s^-, \rho: T^*X \to \mathrm{Hom}(W_s^+, W_s^-))$

On $[-R,R]\times Y$

$\rho(dt): W^+|_{[-R,R]\times Y} \to W^-|_{[-R,R]\times Y}$

$\to$ identify W^- with W^+

$\Rightarrow$ Get $\rho_s : T^*Y \to \mathrm{End}(W^+)$

$\alpha \mapsto \rho(dt)\circ\rho_{\mathbb{X}}(\alpha)$

$\rightsquigarrow \mathfrak{s}_Y = \mathfrak{s}|_Y$

get a spin^c Cliff. mult.

For a solution (A, Φ) on $[-R,R]\times Y$ there is a gauge transf. g s.th. $g(A)$ is in

temporal gauge : $g(A)$ has no dt-component

$\left(\nabla_A s = dt\cdot \frac{ds}{dt} + \nabla_{A(t)} s \right)$

This amounts to solving an ODE along rays $[-R,R] \times \{pt\}$.

If A is in temporal gauge, then (A, Ψ) is a path

$$(A(t), \Psi(t)) \in \mathcal{A}(\Lambda^2 W_3) \times \Gamma(W_3).$$

Config sp. of S/Y

The SW_4 eqns become

$$\boxed{\begin{aligned} \delta\left(\frac{dA}{dt}\right) &= -\rho_3(F_A) + (\Psi(t)\Psi(t)^*)_0 \\ \frac{d\Psi}{dt} &= -D_{A(t)}\Psi(t) \end{aligned}}$$

RHS is RHS of SW_3

$$\Longleftrightarrow \begin{pmatrix} \frac{dA}{dt} \\ \frac{d\Psi}{dt} \end{pmatrix} = -\nabla CSD(A(t), \Psi(t))$$

downward gradient flow equ for

$$CSD(A, \Psi)$$
$$= \frac{1}{2} \int_Y (\hat{A} - \hat{A}_0) \wedge (F_{\hat{A}} + F_{\hat{A}_0})$$
$$- \frac{1}{2} \int_Y \langle \Psi, D_A \Psi \rangle \, vol_{h_Y}$$

If we have a gradient flow equ

$$\dot{x}(t) = -\nabla f(x(t))$$

then $f(x(t))$ decreases along flow lines:

$$\frac{d}{dt} f(x(t)) = df(\dot{x}(t))$$
$$= \langle \nabla f(x(t)), \dot{x}(t) \rangle$$
$$\overset{equ}{=} - \langle \nabla f(x(t)), \nabla f(x(t)) \rangle$$
$$\leq 0$$

So if $(A(t), \Phi(t))$ solves the SW_4 eqns on $[-R,R] \times Y$ then

$CSD(A(t), \Phi(t))$ decreasing

$$\ell_R(A,\Phi) := CSD(A(R), \Phi(R)) - CSD(A(-R), \Phi(-R)) > -C_{\text{indep of } R}$$

$\boxed{\text{Crucial step}}$ ℓ_R is uniformly bounded on the space of solutions by a constant that is independent of R.

(uses a priori estimate and compactness arguments)

Take $R = N \in \mathbb{N}$, consider $N \to \infty$

$$e_N(A_N, \Phi_N)$$

$$= \underbrace{e_{[-N,-N+1]}}_{\substack{\text{drop} \\ \text{between} \\ [-N,-N+1] \ldots\ldots}} + \ldots + e_{[-i,-i+1]} + \ldots + e_{N-1}$$

$$> -C$$

$\Rightarrow$ get a subsequence (N_i) and $[j_i, j_i+1]$ s.th.

$$e_{[j_i, j_i+1]} \longrightarrow 0 \quad (\text{as } i \to \infty)$$

In the limit we get a solution (A, Φ) (up to translation) on $[0,1] \times Y$ $\boxed{\text{with drop in CSD} = 0}$

i.e. $\frac{dA}{dt} = 0$, $\frac{d\Psi}{dt} = 0$

$\Rightarrow$ get a sol^n to the 3-dim'l SW equs.

IV The adjunction ineq.

Thm Suppose $\alpha \in H^2(X;\mathbb{Z})$ is a SW-basic class and $\sigma \in H_2(X;\mathbb{Z})$ with $\sigma\cdot\sigma \geq 0$. Let $\Sigma \hookrightarrow X^4$ rep. σ, i.e. $[\Sigma] = \sigma$. Then

$$\chi_-(\Sigma) \geq \sigma\cdot\sigma + |\langle\alpha,\sigma\rangle|$$

Proof: Let $c_1(\mathfrak{s}) = \alpha$.

$\boxed{\text{Wlog } \Sigma \text{ connected}}$

1st case: $\sigma\cdot\sigma = 0$

$\Rightarrow$ Σ has trivial normal bundle $\cong \Sigma \times D^2$.

We consider a constant curvature metric on Σ with area$(\Sigma)=1$

$$\int_\Sigma \underbrace{K}_{\text{Gauß curvature}} \mathrm{vol} = 2\pi\,\chi(\Sigma)$$

Now $s_\Sigma = 2K$

$$\Rightarrow s_\Sigma = \begin{cases} -4\pi\,\chi_-(\Sigma) & \text{if } g(\Sigma)\geq 1 \\ 4\pi & g(\Sigma)=0 \end{cases}$$

Take product metric on $S^1\times\Sigma$

$$s_{S^1\times\Sigma} = s_\Sigma$$

Let (A,Ψ) be a solution of $SW_3(s|_Y)$

guaranteed by neck-stretching flow

Now

$$|\langle c_1(s), \sigma\rangle| \underset{\text{Weil theory}}{\overset{\text{Chern-}}{=}} \left|\frac{i}{2\pi}\int_{\Sigma} F_{\hat{A}}\right|$$

$$c_1(s) = \left[\frac{i}{2\pi} F_{\hat{A}}\right]$$

$$= \left|\frac{i}{2\pi}\int_{\Sigma \times \{pt\}} F_{\hat{A}}\right|$$

By SW eqn

$$|F_{\hat{A}}| = \frac{1}{\sqrt{2}} \left|(\Phi\Phi^*)_0\right|$$

(because δ multiplies norm by $\sqrt{2}$)

$$= \frac{1}{2}|\Phi|^2 \leq -\frac{1}{2} s_{\Sigma}$$

$$\boxed{(\Phi\Phi^*)_0 = \begin{pmatrix} \frac{1}{2}|\Phi|^2 & 0 \\ 0 & -\frac{1}{2}|\Phi|^2 \end{pmatrix} \Rightarrow |(\Phi\Phi^*)_0| = \frac{1}{\sqrt{2}}|\Phi|^2}$$

$$\Rightarrow |\langle c_1(\mathfrak{s}), \sigma\rangle| \le \frac{1}{2\pi}\sup|F_A| \cdot \mathrm{area}(\Sigma)$$

$$\le \begin{cases} -\frac{1}{2\pi}\, s_\Sigma & \text{if } g(\Sigma) \geq 1 \\ 0 & \text{if } g(\Sigma) = 0 \end{cases}$$

If $\sigma \cdot \sigma =: k > 0$

$\rightsquigarrow$ blow up $X_k = X \# k \cdot \overline{\mathbb{CP}}^2$

$$\alpha_k = \alpha + e_1 + \dots + e_k$$

$$\tilde{\Sigma} := \Sigma \# E_1 \# \dots \# E_k$$

exc. div.

$$\tilde{\Sigma} \cdot \tilde{\Sigma} = 0$$

$\Rightarrow$ apply previous case. ∎